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CONVERGENCE THEOREMS FOR FINITELY ADDITIVE INTEGRALS

By R. C. BUCK

[Received February 20, 1959]

1. Introduction. In a paper which appeared a number of years ago [2], dealing with the contrasting properties of arithmetic and asymptotic density for sets of positive integers, I showed that although the latter was not a measure—since the union of two sets with density need not have density—nevertheless one retained certain additivity properties if one worked modulo sets of zero density. More recently in [3], it was shown that this was not an isolated case but could be subsumed under a more general " additivity theorem " applying to what is termed a generalized asymptotic density. In the present paper, the approach is broadened to the context of function spaces. Instead of a semi-additive set function, we shall be working with linear functionals which are weak limits of measures, and the main result will be an analogue for the Osgood-Lebesgue bounded convergence theorem. At the same tine, we will see that a function space which is initially not σ-complete in its natural ordering may become so if we factor out a properly chosen incomplete subspace.

Section 2 is devoted to a summary of some aspects of the theory of partially ordered real linear spaces which are relevant to the problem, and which motivate the constructions used.

2. Ordered linear spaces. Let V be a real linear space. A partial order is defined in V by selecting a cone P such that $P \cap -P = \{0\}$, and specifying that $x \geqslant y$ if and only if $x - y \in P$. Additional requirements can be imposed on the order relation. The weakest is to ask that every pair of elements in V have at least one upper

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bound; this is equivalent to the requirements that every point in V be a difference of positive elements, and can be achieved (for example) by requiring that P contain an interior point e, which is then called an *order unit* for V. Somewhat stronger is the requirement that V be a *weak lattice*; given x and y in V, let $U(x, y)$ be the set of all $t \in V$ with $t \geqslant x$ and $t \geqslant y$; then, we ask that every two elements in $U(x, y)$ have a lower bound in $U(x, y)$. There is an alternative characterization for this property, used in this form by F. Riesz (see also [7], [1]).

LEMMA. *V is a weak lattice if and only if given three positive elements x, y, z with $0 \leqslant z \leqslant x+y$, one may always write $z = x_0 + y_0$, where $0 \leqslant x_0 \leqslant x$ and $0 \leqslant y_0 \leqslant y$.*

PROOF. Assuming that x, y and z are so related, $z - y$ and 0 are both lower bounds for x and z. Assuming the weak lattice property, there is an element u with $0 \leqslant u \leqslant x$ and $z - y \leqslant u \leqslant z$. Setting $x_0 = u$ and $y_0 = z - u$ gives the desired decomposition of z. Conversely, if V has the (Riesz) decomposition property, suppose we are given two lower bounds a and b for two elements x and y. Set $u = x - a$, $v = y - a$ and $w = b - a$. Since $u + (v - w) \geqslant v$, with all three positive, we can split v as $s + t$ with $0 \leqslant s \leqslant u$ and $0 \leqslant t \leqslant v - w$. Then, $c = a + s$ is a lower bound for x and y which is an upper bound for both a and b.

A still stronger requirement is that V be in fact a lattice; here, we ask that each of the sets $U(x, y)$ have the form $I(c) = \{\text{all } z \in V \text{ with } z \geqslant c\}$, for some element c, namely the least upper bound $x \cup y$ of x and y. It may be shown that a vector lattice is always distributive (see [1]).

Let W be a subspace of V. Then W will inherit the order relation from V, but may fail to share other more special features of V. It is convenient to assume that W is *interval closed*, so that whenever $u \leqslant x \leqslant v$ with u and v in W, we also have $x \in W$; this condition characterizes those subspaces of V which are the kernels (null spaces) of order preserving linear transformations on V, and permits

the introduction of a natural ordering into V/W. Let W_0 denote the subspace of W spanned by the positive elements in W. Then, one readily sees that in this case, if V is a weak lattice, so is W_0, and if V is a lattice, so is W_0; moreover, in each case W_0 is the largest subspace of W having this property. The assumption that W was interval closed is crucial for this. The space $C[R]$ of all bounded continuous real-valued functions on the line R is a vector lattice in the usual ordering ; the subspace of infinitely differentiable functions is only a weak lattice.

One may also distinguish various degrees of order completeness. The weakest (here to be called σ-completeness) asks that every bounded monotone sequence $x_1 \leqslant x_2 \leqslant x_3 \leqslant \ldots$ have a least upper bound. If V is a lattice, the monotonicity restriction can be removed; more generally, one might require that every bounded set with a specified cardinal number should have a least upper bound. In general, one expects that if V and W both share a property, then V/W will also. The following is more or less typical. (Since this is not to be found in [1], I append a proof.)

Lemma 2. *Let V be a vector lattice, and W an interval-closed subspace which is also a sublattice. If V and W are both σ-complete, then so is V/W.*

Proof. Suppose $x_n \in V$ with $\{x_1, x_2, \ldots\}$ bounded above by b modulo W. We can then choose $c_n \in W$ with $x_n \leqslant b + c_n$. Let y be the least upper bound in V for the countable bounded set $\{x_1', x_2', \ldots\}$, where $x_n' = x_n - c_n$. We show that y is the least upper bound for $\{x_1, x_2, \ldots\}$ modulo W. Suppose y' is another upper bound ; then there exist $a_n \in W$ so that $y' + a_n \geqslant x_n$. Set $a_n' = (a_n - c_n) \cap |b|$, noting that this lies in W. Then, $\{a_1', a_2', \ldots\}$ is bounded from above in V, and by assumption, its least upper bound a' lies in W. One notes that $y' + a' \geqslant x_n'$ for all n, so that $y' + a' \geqslant y$, and $y' \geqslant y$ modulo W.

More than this can sometimes happen; there is a familiar example in which the quotient lattice is better behaved than the original. The lattice of measurable sets is σ-complete, and so is the sublattice

of sets of measure zero, but the quotient lattice is *unrestrictedly* complete. This suggests the possibility that one might have V/W σ-complete, when V and W themselves are not; this is the key to the work of the next section.

3. The convergence theorem. Although the main results could be formulated more generally, we prefer to work in the framework of a fixed partially ordered linear space of functions. Let Ω be a locally compact space, and let $C[\Omega]$ be the space of bounded real valued continuous functions on Ω. One says $f \geqslant 0$ if $f(p) \geqslant 0$ for all $p \in \Omega$. $C[\Omega]$ is a vector lattice with the constant function 1 as order unit. Let L_n be a sequence of positive linear functionals, normalized by the condition $L_n(1) = 1$. We shall later impose a further condition on the sequence $\{L_n\}$. For any $f \in C[\Omega]$, we may define two numbers

$$L^*(f) = \limsup L_n(f)$$
$$L_*(f) = \liminf L_n(f).$$

The functionals L^* and L_* are not linear; let $V \subset C[\Omega]$ be the set of all f for which $L^*(f) = L_*(f)$, and denote this common value by $\mathscr{L}(f)$. Then, V is a linear subspace of $C[\Omega]$, and $\mathscr{L}$ is a positive linear functional on V. Let $W \subset V$ be the null space of $\mathscr{L}$, and let W_0 be the subspace generated by the positive elements in W.

An example will help to clarify these constructions. Let Ω be the positive integers with discrete topology, so that $C[\Omega]$ is the space of bounded real sequences. Choose L_n as the linear functional

$$L_n(f) = \frac{f(1) + f(2) + \dots + f(n)}{n}.$$

Then, V is the space of $(C,1)$ summable sequences, $\mathscr{L}$ is the Cesàro limit, W is the space of bounded sequence summable to zero and W_0 is the space of sequences that can be expressed as the difference of two positive $(C, 1)$ null sequences.

The space V inherits a partial ordering from $C[\Omega]$, although it will not be a lattice. The subspaces W and W_0 are interval closed. We may therefore form the quotient space V/W_0. One of our main

results is the assertion that this space is σ-complete; the second assorts that the functional $\mathscr{L}$ is order continuous on V/W_0. We need a further restriction on the functionals L_k. We assume that there is a special sequence $\psi_n \in W$ with $0 \leqslant \psi_1 \leqslant \psi_2 \leqslant \ldots 1$ and $\psi_n \psi_{n+1} = \psi_n$, such that $\psi_n \to 1$ in the sense that $\lim_{n\to\infty} L_k(f\psi_n) = L_k(f)$ for each $k = 1, 2, \ldots$. In the example above, ψ_n can be chosen as the sequence $(1, 1, 1, \ldots, 1, 0, 0, 0, \ldots)$. More generally, one would select an exhaustion of Ω by compact sets, and construct ψ_n with compact support such that for any f, $\psi_n f$ converges to f uniformly on compact sets. If each functional L_k is continuous in the strict topology β (see [5]) then the second requirement is obeyed. It is also convenient to introduce a third subspace $W_\infty \subset W_0$, by saying that $f \in W_\infty$ if and only if there are numbers n and B such that $-B\psi_n \leqslant f \leqslant B\psi_n$. In the example above, W_∞ is the space of finitely non-zero sequences; in general, W_∞ is an adequate replacement for the space of functions with compact support.

Suppose now that we have a sequence $f_n \in V$ with $0 \leqslant f_1 \leqslant f_2 \leqslant f_3 \ldots \leqslant 1$, and suppose that F is the least upper bound of the f_n. Does it follow that $\mathscr{L}(F) = \lim \mathscr{L}(f_n)$? A negative answer is supplied by the functions ψ_n themselves since here $F = 1$ and $\mathscr{L}(\psi_n) = 0$ for each n. It is natural to hope that factoring out all or part of W would improve things.

Theorem 1. *Let $f_n \in V$ with $0 \leqslant f_1 \leqslant f_2 \leqslant \ldots \leqslant 1$. Then there is a function $F \in V$ such that $f_n \leqslant F$ modulo W_∞ for all n, and such that*

$$\mathscr{L}(F) = \lim \mathscr{L}(f_n).$$

Moreover, F is the (unique) least upper bound of the sequence $\{f_n\}$ modulo W_0.

Proof. Since $\lim_{m\to\infty} L_k(\psi_m) = 1$, we can choose a monotone sequence $s(k)$ such that if $m \geqslant s(k)$, $L_k(\psi_m) \geqslant 1 - (1/k)$. Suppose now that F is any function in $C[\Omega]$ with $0 \leqslant F \leqslant 1$. Then, for any $p \in \Omega$,

$$F(p) \leqslant F(p)\,\psi_m(p) + 1 - \psi_m(p).$$

and

$$L_k(F) \leqslant L_k(F\,\psi_m) + (1/k), k = 1, 2, \dots, m \geqslant s(k). \tag{1}$$

Since $f_n \in V$, $\lim_{k\to\infty} L_k(f_n) = \mathscr{L}(f_n)$, and we can choose an increasing sequence r_n so that

$$L_k(f_n) \leqslant \mathscr{L}(f_n) + (1/n), \quad k \geqslant r_n. \tag{2}$$

Set $\lambda_n = s(r_{n+2})$, and then construct a special function F by

$$F = f_1\,\psi_{\lambda_1} + \sum_2^\infty f_n\,\{\psi_{\lambda_n} - \psi_{\lambda_{n-1}}\}.$$

(Effectively, we have here used the familiar partition-of-unity technique.)

Using the fact that $\psi_n\,\psi_{n+r} = \psi_n$, we see easily that

$$F\,\psi_{\lambda_{n-1}} = f_1\,\psi_{\lambda_1} + \sum_2^{n-2} f_m\,\{\psi_{\lambda_m} - \psi_{\lambda_{m-1}}\} + f_{n-1}\,\{\psi^2_{\lambda_{n-1}} - \psi_{\lambda_{n-2}}\} + \\ + f_n\,\{\psi_{\lambda_{n-1}} - \psi^2_{\lambda_{n-1}}\} \leqslant f_n\,\psi_{\lambda_{n-1}} \leqslant f_n.$$

Likewise, $f_n \leqslant F + \psi_{\lambda_n}$; note that this shows that F is an upper bound for the sequence $\{f_n\}$ modulo W_∞.

Given any k, choose n so that $r_n \leqslant k < r_{n+1}$. Then, $\lambda_{n-1} = s(r_{n+1}) > s(k)$, so that by (1)

$$L_k(F) \leqslant L_k(F\,\psi_{\lambda_{n-1}}) + (1/k).$$

Since $F\,\psi_{\lambda_{n-1}} \leqslant f_n$, we may use (2) to obtain

$$L_k(F) \leqslant L_k(f_n) + (1/k) \leqslant \mathscr{L}(f_n) + (1/n) + (1/k).$$

Let $d = \lim \mathscr{L}(f_n)$. Then, allowing k (and with it n) to increase, we have shown that $L^*(F) \leqslant d$. However, since $(f_n) \leqslant F + \psi_{\lambda_n}$, we may conclude that $L_k(f_n) \leqslant L_k(F) + L_k(\psi_{\lambda_n})$, and thus $\mathfrak{L}(f_n) \leqslant L^*(F)$. Letting n increase, we have $d \leqslant L^*(F)$. Combining these, we have $F \in V$ and $\mathscr{L}(F) = d = \lim L(f_n)$.

It only remains to prove that F is minimal, modulo W_0, among all upper bounds for the sequence $\{f_n\}$. Suppose that F' were

another candidate. Form $F'' = F \cap F'$ in $C[\Omega]$. Then, F'' too is an upper bound for $\{f_n\}$ modulo W_0. Since $F'' \leqslant F$, we see that for any n, $\mathscr{L}(f_n) \leqslant L_*(F'') \leqslant L^*(F'') \leqslant \mathscr{L}(F) = d$, and letting n increase, $F'' \in V$ with $\mathscr{L}(F'') = d$. Put $g = F - F''$; since $g \geqslant 0$ and $\mathscr{L}(g) = 0$, $g \in W_0$. Finally, $F'' \leqslant F'$ so that $F \leqslant F' + g$ and $F \leqslant F'$ modulo W_0.

It is now easy to specialize this result to obtain the additivity theorem for asymptotic densities; as in [3], let $\{\mu_k\}$ be a sequence of measures on Ω and $K(n)$ an increasing sequence of measurable sets whose union is all of Ω. Choose ψ_n to be the characteristic function of $K(n)$, and define the functionals L_k by

$$L_k(f) = \int_{\Omega} f(p)\, d\mu_k(p).$$

If f is the characteristic function of a set $A \subset \Omega$, then $f \in V$ if and only if A has a density, and $\mathscr{L}(f)$ is exactly $D(A)$. Since a characteristic function is non-negative, W_0 contains all the characteristic functions of the sets of zero density. Finally, if $A_1 \subset A_2 \subset \ldots$ is an increasing sequence of densible subsets of Ω, we can apply Theorem 1 to their characteristic functions to obtain the function F; it is easily seen that F is also a characteristic function (modulo W_0), so that we have found a set A which is the least upper bound of the sequence $\{A_n\}$ modulo sets of zero density, and such that $D(A) = \lim D(A_n)$. (This paragraph may be taken as replacing the original proof of the additivity theorem in [3] which contained a number of confusing misprints; it should also be noticed that we have not needed to assume that the measures μ_k had bounded support.)

4. Characterization of W_0. Since continuity of $\mathscr{L}$ was restored by treating V modulo W_0, there is some interest in finding an alternative description of W_0. We must first introduce $\overline{W}_\infty$, the order closure of W_∞; this is the analogue in our framework for the space of functions that " vanish at infinity ". Specifically,

$f \in \overline{W}_\infty$ if and only if given any $\delta > 0$, there is a function $h \in W_\infty$ with $-\delta - h \leqslant f \leqslant h + \delta$. Clearly, $W_\infty \subset \overline{W}_\infty \subset W_0$.

THEOREM 2. *$F \in W_0$ if and only if there is a set S of zero density such that $F(p) \to 0$ "at infinity" in $\Omega - S$.*

If ϕ is the characteristic function of S, this amounts to saying that $\{1 - \phi\} F \in W_\infty$.

PROOF. We may suppose that we are concerned with a function F obeying $0 \leqslant F \leqslant 1$. Suppose $\mathscr{L}(F) = 0$. Let A_n be the set of all p with $F(p) \geqslant 1/n$, and let ϕ_n be its characteristic function. Then, one has

$$(1/n)\, \phi_n \leqslant F \leqslant \phi_n + (1/n).$$

In particular, $\mathscr{L}(\phi_n) = 0$ so that $D(A_n) = 0$. By the additivity theorem, construct S with $D(S) = 0$ such that $A_n \subset S$ modulo W_∞. With ϕ as the characteristic function of S, this means that here are functions $h_n \in W_\infty$ with $\phi_n \leqslant \phi + h_n$. We must then have $0 \leqslant F \leqslant \phi + h_n + (1/n)$, showing that F has the stated structure. Conversely, if F has this structure, one readily sees that $\mathscr{L}(F) = 0$.

Specialized to the sequence-space illustration used earlier, this becomes the statement that a bounded non-negative sequence is Cesàro summable to 0 if and only if it coincides with a null sequence except for a set of indices of zero asymptotic density. (This is quite relevant to the theory of *strong* or *absolute* summability and explains some of the peculiarities of these methods ; in this connection, see also [7], [8]).

As a continuation of the general program of relating measure theory to density theory, it would be of some interest to study the functions f on Ω which have the property that all of the spectral sets $A_\lambda = \{\text{all } p \in \Omega \text{ with } f(p) > \lambda\}$ are sets having density ; for these, one could at once formulate the notion of "integration with respect to a density".

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University of Wisconsin
and
Stanford University

$\mathfrak{p}$-ADIC FORMS

By B. J. BIRCH and D. J. LEWIS

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1. It has long been conjectured that a homogeneous polynomial in n variables of a given fixed degree d over a $\mathfrak{p}$-adic number field has a non-trivial zero in that field provided $n \geqslant d^2 + 1$. It was shown by Hasse [7] that this is true in the quadratic case $d = 2$, and the cubic case $d = 3$ has been treated independently by Demyanov [5] and by Lewis [10], and more recently by Davenport [4]; another interesting approach to the cubic case has been suggested by Springer [20], and although his approach cannot be directly extended to higher degrees, our forms ${}^jF^*$ bear a resemblance to his residue class forms. In the general case, Brauer [0] has shown that there is a function $\psi(d)$ such that a $\mathfrak{p}$-adic form of degree d in more than $\psi(d)$ variables always has a zero.

In this paper we investigate the only known approach to the problem for higher degrees; our analysis shows that the conjecture is also true for $d = 5$, provided the residue class field is large; we also obtain some more general results (see § 6).

In § 2 we derive some results on equations over finite fields, as immediate corollaries of a theorem of Lang and Weil [8]; the result we really need is Corollary 2, but while we are about it we enunciate some other results which do not seem to be generally known, notably Corollary 3. In §3 and §4 are two lemmas which tell us how zeros of a $\mathfrak{p}$-adic polynomial are derived from zeros in its residue class field; much of the hardest work in the paper is in the proof of Lemma B, which essentially describes a way of normalizing a $\mathfrak{p}$-adic form. In § 5 we complete the proof of our theorem on quintic forms.

2. In [8], Lang and Weil proved the following important theorem:

THEOREM 1. *There is a constant $A(m, d, r)$ such that if V is an absolutely irreducible variety of dimension r and degree d in m-dimensional projective space P^m, defined over the finite field k_q with q elements, and if M is the number of points of V in k_q, then*

$$|M - q^r| \leqslant (d-1)(d-2)q^{r-1/2} + A(m, d, r)q^{r-1}.$$

They obtain this result as an easy consequence of Weil's deep estimate for the number of points on algebraic curves. We need the particular case of the theorem where $r = m - 1 = n - 2$; since V is a projective variety, each point of V gives $(q - 1)$ zeros of the corresponding absolutely irreducible polynomial in n variables, and we have

COROLLARY 1. *If N is the number of zeros in k_q of an absolutely irreducible homogeneous polynomial F of degree d in n variables, F having its coefficients in k_q, then*

$$|N - q^{n-1}| < A(n, d)\, q^{n-3/2},$$

where $A(n, d)$ depends on n, d only. (Unfortunately, A is hard to estimate for $n > 3$.)

Also implicit in the paper of Lang and Weil are the two following results, the first of which is a particular case of their Lemma 1, and the second of which follows from their Lemmas 1 and 2.

LEMMA 1. *If N is the number of zeros in k_q of a (not necessarily irreducible) polynomial of degree d in n variables, then*

$$N < B(n, d)\, q^{n-1},$$

where B depends only on n and d.

A zero $y = (y_1, \dots, y_n)$ of a polynomial $f(x_1, \dots, x_n)$ is said to be *singular* if $\dfrac{\partial f}{\partial x_1} = \dfrac{\partial f}{\partial x_2} = \dots = \dfrac{\partial f}{\partial x_n} = 0$ at y. We have

LEMMA 2. *If F is a form of degree d in n variables, with no squared factors, and N^* is the number of singular zeros of F in k_q, then*

$$N^* < C(n, d)q^{n-2},$$

where C depends only on n and d.

We can now deduce

COROLLARY 2. *Let F be a form of degree d in n variables, over a finite field k_q. Then, so long as $q \geqslant D(n, d)$, F has more than $\frac{1}{2} q^{n-1}$ non-singular zeros unless it is reducible over some extension of k_q. Conversely, if F is irreducible over k_q but not absolutely irreducible, then all the zeros of F are singular.*

PROOF. The first part is immediate from Corollary 1 and Lemma 2, since $N - N^* > \frac{1}{2} q^{n-1}$ so long as $\frac{1}{2} q^{n-1} > A(n, d)\, q^{n-3/2} + C(n, d)\, q^{n-2}$.

As for the second part, if F is irreducible over $k = k_q$ but not absolutely irreducible, there is a normal extension k' of k in which F splits into conjugate factors, $F = F_1 \dots F_h$ say. Now $F_1, \dots, F_h$ will have coefficients in k' and will be conjugate over k; so if y is a zero of F in k, it is a zero of one of the F_i and so by conjugacy of all of them. Thus y is an h-fold and so a singular zero of F.

This proves the corollary, which generalizes the results of Lewis [11]. The example given at the end of [11] shows that the condition $q \geqslant D(n, d)$ is essential.

REMARK 1. For $d = 3$, the condition that $q \geqslant D(n, 3)$ may be dropped. Indeed, any field k_q may be made large enough by successive quadratic extensions, and the result follows from the fact used by Lewis [12] that if a cubic form G defined over a field k has a non-singular zero in a quadratic extension $k(\theta)$, it already has a non-singular zero in k. The following simple justification was pointed out to one of us by H. P. F. Swinnerton-Dyer. Let V be the surface $G = 0$, and let P be a simple point of V with coordinates in $k(\theta)$; if P has coordinates in k, there is nothing to prove. Otherwise, the conjugate $\bar{P}$ of P over k is also a simple point of V, and P, $\bar{P}$ are distinct; the line $P\bar{P}$ now meets V in a third point, Q say, and Q will be a simple point of V with coordinates in k.

REMARK 2. Suppose that a field k is an infinite algebraic extension of a Galois field—then any absolutely irreducible polynomial form over k has a non-singular zero in k. (Indeed, if k_q is a large enough Galois field contained in k and containing all the coefficients of F, the corollary tells us that F has a non-singular zero in k_q.) This property is apparently stronger than 'quasi-algebraic closure' (see Lang [9])—it should be interesting to know what other fields have it.

LEMMA 3. *Let F be a homogeneous form of degree d in n variables $x_1, x_2, \ldots, x_n$ over a field k. Suppose F is not a power of another form (that is, there is no form G over $\bar{k}$, the algebraic closure of k, such that $F = G^e$, with $e > 1$). Then the form $F - x_0^d$ is absolutely irreducible.*

PROOF. Consider the form $(x_0^d - ct^d)$, with c in $\bar{k}$; this is equal to $\prod_{j=1}^{d} (x_0 - \gamma \rho_j t)$, where $\gamma^d = c$ and ρ_j runs through the dth roots of 1; so if $x_0^d - ct^d$ factorises as $\left(\sum_{i=0}^{r} a_i x_0^{r-i} t^i\right) \left(\sum_{j=0}^{d-r} b_j x_0^{d-r-j} t^j\right)$, with $a_0 = b_0 = 1$ and the a_i, b_j in $\bar{k}$, then a_r and b_{d-r} are both products of $\gamma \rho_i$'s, and a_r^{d-r}/b_{d-r}^r is a dth root of 1.

Now suppose that $x_0^d - F$ factorises as $\left(\sum_{i=0}^{r} A_i x_0^{r-i}\right) \left(\sum_{j=0}^{d-r} B_j x_0^{d-n-j}\right)$, where the A_i, B_j are polynomials in $x_1, \ldots, x_n$ over $\bar{k}$, and $A_0 = B_0 = 1$. Then $A_r B_{d-r} = -F$. Specialise $x_1, \ldots, x_n$ to $\xi_1 t, \ldots, \xi_n t$, where $\xi_1, \ldots, \xi_n$ are particular values in $\bar{k}$; then the factorisation of $x_0^d - F$ induces a factorisation of $x_0^d - ct^d$, where $c = F(\xi)$. By the remark above, $[A_r(\xi)]^{d-r}/[B_{d-r}(\xi)]^r$ must be a dth root of 1; that is, this ratio has only d values as $\xi_1, \ldots, \xi_n$ run through the whole of $\bar{k}$. This is impossible unless $[A_r(x)]^{d-r}/[B_{d-r}(x)]^r$ is a constant, and then, by the unique factorization theorem for polynomials, $A_r(x)$ and $B_{d-r}(x)$ are proportional to powers of the same polynomial, and so $-F = A_r B_{d-r}$ is a perfect power.

COROLLARY 3. *Let F be a homogeneous form of degree d in n variables over k_q, not a power of another form. Then, so long as*

$q \geqslant E(n, d)$, F is universal—that is, for any a in k_q there are $y_1, \ldots, y_n$ in k_q such that $F(y_1, y_2, \ldots, y_n) = a$.

PROOF. If $a = 0$, we can simply take $y_i = 0$. Suppose then that $a \neq 0$. Consider the form $F - ax_0^d$. If F is not a power of another form, neither is $a^{-1}F$ and thus by Lemma 3, $F - ax_0^d$ is absolutely irreducible; so by Corollary 1, $F(x_1, \ldots, x_n) - ax_0^d = 0$ has N solutions, where $|N - q^n| < A(n+1, d)q^{n-1/2}$. By Lemma 1, $F(x_1, \ldots, x_n) = 0$ has at most $B(n, d)q^{n-1}$ solutions, so for $q \geqslant E(n, d)$, $F - ax_0^d = 0$ has a solution with $x_0 \neq 0$. Then $F(x_1/x_0, \ldots, x_n/x_0) = a$.

This corollary may be regarded as a sharpening of a result of Carlitz [1], which asserts that a form over a large enough field k_q which represents only hth powers is itself an hth power (this theorem is trivially false for small q). So long as q is large enough, our corollary gives a complete answer to the question, 'which forms are universal?'. However, for small q very little seems to be known for forms which do not happen to be diagonal, beyond what can be deduced trivially from the well-known theorem of Chevalley [2]. Results for cubics have been obtained by Marshall Hall, (see Selmer [18, p. 220]), and by Piehler [17].

3. As in Lewis' paper [10], K denotes a field complete under a discrete non-archimedean valuation v, $\mathfrak{o}$ is the ring of integers of K, π a local prime of K, and $\mathfrak{p} = \pi\mathfrak{o}$ the prime ideal generated by π. We suppose that the residue class field $\mathfrak{o}/\mathfrak{p}$, which we denote by k, has q elements. The residue class containing the integer a is denoted by a^*, and if $F(\mathfrak{x}) = F(x_1, \ldots, x_n)$ is a polynomial over $\mathfrak{o}$, $F^*(\mathfrak{x})$ has the natural meaning.

K^n denotes the n-dimensional vector space over K; a vector of K^n is *primitive* if it has integral coordinates, at least one of which is a unit. The primitive vectors are a compact subset of K^n. Clearly, the mapping (*) maps the primitive vectors of K^n onto the non-zero vectors of k^n. A basis for K^n is *admissible* if it consists of integral vectors, and if the mapping (*) takes it into a basis for k^n (we deduce that the vectors of an admissible basis are primitive). An *admissible*

transformation takes an admissible basis into an admissible basis, it is simply a linear transformation whose matrix is integral unimodular. Clearly these transformations form a multiplicative group and the column and row vectors of the matrices representing these transformations are primitive; consequently the group of admissible transformations is compact.

In [10], it was shown how a solution of $F(\mathfrak{x}) = 0$ may be deduced from a non-singular solution of $F^*(\mathfrak{x}) = 0$. This is done by the so-called Newton approximation process :

LEMMA A. *Let $F(\mathfrak{x})$ be a homogeneous polynomial with coefficients in $\mathfrak{O}$, and let b be any element of $\mathfrak{O}$. Suppose that $\mathfrak{z}^*$ is a non-zero vector of k^n such that $F^*(\mathfrak{z}^*) = b^*$ and $\dfrac{\partial F}{\partial x_r}(\mathfrak{z}^*) \neq 0$ for at least one index r. Then there is a primitive vector $\mathfrak{y}$ of K^n such that $F(\mathfrak{y}) = b$ and $\mathfrak{y}^* = \mathfrak{z}^*$.*

(This is simply Lemma 1 of [10].)

COROLLARY 4. *Let $F(\mathfrak{x})$ be a homogeneous polynomial of degree d with coefficients in $\mathfrak{O}$, such that $F^*(\mathfrak{x})$ is absoluely irreducible. Then, so long as q (the order of the residue class field) is large enough compared to n and d, and so long as q is prime to d, $F(\mathfrak{x})$ is universal.*

PROOF. By Corollary 3, we can solve $F^*(\mathfrak{x}) = b^*$ for any $b^* \neq 0$ of k. Further, since $\sum_{r=1}^{n} x_r \dfrac{\partial F^*}{\partial x_r} = d^* F^*$, we know $\dfrac{\partial F}{\partial x_r} \neq 0$ for at least one r. On the other hand, by Corollary 2 there is a non-singular solution of $F^*(\mathfrak{x}) = 0$. The result now follows immediately by Lemma A.

This corollary is an extension to general forms of various results which have been proved for diagonal forms, see for instance Cohen [3]. Clearly, by following our argument in greater detail, it is possible to obtain good estimates for the number of solutions of, say, $F(\mathfrak{x}) \equiv b \pmod{\mathfrak{p}^t}$. The conditions on q are essential, and, as in Corollary 3, allow us to avoid the main difficulties of the problem

rather shamelessly! For a glimpse of the difficulties which may arise when q is not prime to d, see Gray [6], Lewis [13], or O'Meara [14], [15], [16].

4. Until recently, except in the cubic and quadratic cases, very little was known about forms over finite fields having only singular zeros; now that this omission has been repaired by Corollary 2, we know that if q is large enough the forms without non-singular zeros are of easily recognised types. To go any further, we must show that if $F(\mathfrak{x})$ is a form over K, there is a form $G(\mathfrak{x})$ 'equivalent' to F such that G^* is of a nice type. This does not seem to be an easy matter; Lemma B, to which proof this paragraph is devoted, is the best that we have been able to manage.

Some notation is needed. Let $H(\mathfrak{x})$ denote a form in $x_1, x_2, \ldots, x_n$. As usual, let $\mathfrak{x} = T\mathfrak{y}$ denote the linear transformation $x_r = \sum_{j=1}^{n} t_{rj}\, y_j (r = 1, 2, \ldots, n)$, then $H(T\mathfrak{y})$ is a new form in $\mathfrak{y}$, denoted by $H_T(\mathfrak{y})$; $H(T\mathfrak{y}) \equiv H_T(\mathfrak{y})$. We shall use this notation consistently; observe that $(H_T)_S = H_{(TS)}$. Let $\gamma(H)$ denote the number of variables that occur in monomials of H with non-zero coefficient. The order of H with respect to a field L, written $o_L(H)$, is defined by

$$o_L(H) = \min_T \gamma(H_T),$$

where T runs through all non-singular linear transformations with coefficients in the field L; clearly if B is a non-singular linear transformation with coefficients in L, $o_L(H) = o_L(H_B)$. When we speak of 'the order of H' and when we write '$o(H)$', we shall mean order with respect to K or k as the case may be. A form has order one if and only if it is a constant multiple of a power of a linear form; and in general, if $H(x_1, x_2, \ldots, x_n)$ has order m, we can write

$$H(x_1, \ldots, x_n) = G(L_1, \ldots, L_m), \tag{1}$$

where $L_1, \ldots, L_m$ are linear forms in $x_1, \ldots, x_n$, say $L_j = \sum_{r=1}^{n} L_{jr}\, x_r$, with rank $\|L_{jr}\| = m$ and of course each L_{jr} in K; and G is a form over K which cannot be transformed by a linear transformation into

a form involving fewer than m variables. When K is the quotient field of $\mathfrak{o}$, there is no loss in assuming the L_{jr} to be in $\mathfrak{o}$.

It may be shown without real difficulty that $o_L(H)$ is independent of the field L, so long as L is perfect and contains all the coefficients of H; we do not need this remark, so will not give the proof.

Let $|H|$ denote the largest integer t such that π^t divides every coefficient of H; that is, $|H|$ is the order of the content of H. Clearly $|H| = \min_{M \in H} |M|$, where M runs through the monomials of H and $|M|$ is the highest power of π dividing the coefficient of M.

LEMMA B. *Let $F(\mathfrak{x})$ be a form of degree d in n variables $x_1, \ldots, x_n$ with coefficients in K, suppose that $o(F) = n$, and suppose that $n \geqslant (u-1)d+1$. Then there is a non-singular integral linear transformation, say $\mathfrak{x} = T\mathfrak{y}$, such that*

$$F_T(\mathfrak{y}) = \pi^{|F_T|} H(\mathfrak{y}) \tag{2}$$

with

$$o(H^*) \geqslant u. \tag{3}$$

In the proof of Lemma B, we will operate on F by transformations of two types. In the first place, we will use the compact group of admissible transformations, as defined at the start of § 3. Secondly, we will use *rakes*—these are simply diagonal transformations $F_R(x_1, \ldots, x_n) = F(\pi^{-\gamma_1} x_1, \ldots, \pi^{-\gamma_n} x_n)$, which we use to 'pull the variables down' to where they are useful. Define the *power* of the rake by $\|R\| = \sum_{r=1}^{n} \gamma_r$ (formally $\|R\| = -\log_\pi (\det R)$); call R a *positive* rake if every $\gamma_r \geqslant 0$, and say that R is a *rake for F* if F_R has integral coefficients. The set of all rakes forms a group, and the product of two positive rakes is positive.

LEMMA 4. *If F is a form with integral coefficients which involves all of $x_1, \ldots, x_n$, i.e. $\gamma(F) = n$, then there is a positive rake R for F such that for each r,*

$$F_R(x_1, \ldots, \pi^{-1} x_r, \ldots, x_n)$$

does not have integer coefficients.

Indeed, we may choose γ_1 so that $F(\pi^{-\gamma_1} x_1, x_2, \dots, x_n)$ has integer coefficients but $F(\pi^{-\gamma_1-1} x_1, x_2, \dots, x_n)$ does not. Then we choose γ_2 so that $F(\pi^{-\gamma_1} x_1, \pi^{-\gamma_2} x_2, x_3, \dots, x_n)$ has integer coefficients but $F(\pi^{-\gamma} x_1, \pi^{-\gamma_2-1} x_2, x_3, \dots, x_n)$ does not, and so on.

A positive rake satisfying the conclusion of Lemma 4 is said to be *maximal* for F. Given on F, there may be several positive rakes which are maximal for F. R is maximal for F if and only if I, the identity transformation, is the only positive rake for F_R.

LEMMA 5. *If I is the only positive rake for F, then for each r there is a monomial M of F such that $|M| <$ power of x_r in M.*

The proof is obvious.

If I is the only positive rake for F, we can make an assignment of the variables $x_1, x_2, \dots, x_n$ to d disjoint subsets or *batches* $\mathfrak{B}_0, \mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_{d-1}$: Let $\mathfrak{B}_0$ consist of those variables appearing in F^*. Assuming $\mathfrak{B}_0, \mathfrak{B}_1, \dots, \mathfrak{B}_{k-1}$ have been defined, form

$$^kF(x_1, x_2, \dots, x_n) = \pi^{-k} F(\pi^{\gamma_{k1}} x_1, \pi^{\gamma_{k2}} x_2, \dots, \pi^{\gamma_{kn}} x_n), \tag{4}$$

where $\gamma_{kr} = 1$ if x_r is in $\bigcup_{j=0}^{k-1} \mathfrak{B}_j$, and $\gamma_{kr} = 0$ otherwise. By construction the ^kF's have integral coefficients. (This is not quite trivial.) Let $\mathfrak{B}_k$ consist of those x_r which appear in monomials of $^kF^*$ and which have not previously been assigned. By construction the $\mathfrak{B}_j$'s are disjoint.

Assume that x_r is not assigned to some $\mathfrak{B}_j$. By Lemma 5 there is a monomial, say M, of F such that $|M| < \mu(M)$, the power of x_r in M. Let ^kM denote the image of M under (4). Since ^kF has integral coefficients, $|^kM| \geqslant 0$ for each k; and since x_r is in no $\mathfrak{B}_k$, $|^kM| > 0$ for all k and in particular $|^{d-1}M| > 0$. But

$$\begin{aligned} |^{d-1}M| &= -(d-1) + \sum_{x_j \in M} \gamma_{d-1,j} + |M| \\ &\leqslant -(d-1) + (d - \mu(M)) + |M| \leqslant 0, \end{aligned}$$

since $\gamma_{d-1,j} = 1$ for at most $(d - \mu(M))$ variables appearing in M; and this is a contradiction. Hence every variable is assigned to some $\mathfrak{B}_j$.

Write n_j for the number of variables in $\mathfrak{B}_j$; then $\sum_{j=0}^{d-1} n_j = n$.

Applying these results to F_R, when R is maximal for F, one obtains

LEMMA 6. *To each positive rake R, which is maximal for F, there is associated an assignment of the variables $x_1, x_2, \ldots, x_n$ to d disjoint batches $\mathfrak{B}_0, \mathfrak{B}_1, \ldots, \mathfrak{B}_{d-1}$ such that if*

$$^jF_R(x_1, x_2, \ldots, x_n) = \pi^{-j}F_R(\pi^{\gamma_{j1}}x_1, \pi^{\gamma_{j2}}x_2, \ldots, \pi^{\gamma_{jn}}x_n),$$

where $\gamma_{jr} = 1$ if x_r is in $\bigcup_{i=0}^{j-1} \mathfrak{B}_i$ and $\gamma_{jr} = 0$ otherwise, then each $^jF_R(j = 0, 1, \ldots, d-1)$ has integer coefficients and $^jF_R^*$ involves all of the variables of $\mathfrak{B}_j$ and no variables from $\mathfrak{B}_k$ with $k > j$.

LEMMA 7. *Let F be a form of degree d in n variables, with integral coefficients, and such that $o(F) = n$. Then there is a constant K_0 (depending only on F) such that $\max(\gamma_r) \leqslant K_0$ for all admissible transformations B, and all positive rakes for F_B.*

PROOF. If R is a positive rake for F_B, then

$$F_{BR}(x_1, \ldots, x_n) = F_B(\pi^{-\gamma_1}x_1, \ldots, \pi^{-\gamma_n}x_n)$$

has integer coefficients; so if $M = c.x_r.M'$ (M' being just a product of $d-1$ x's) is any monomial of F_B which involves x_r, $c\pi^{-\gamma_r}$ is an integer; that is, every term involving x_r in F_B is congruent to zero mod π^{γ_r}.

If now γ_r is not bounded, we can find a sequence $\{B^{(t)}\}$ of admissible transformations such that the coefficient of every term of $F_{B^{(t)}}$ involving x_r tends to zero as $t \to \infty$. Since the admissible transformations are compact this sequence will have at least one limit, say B^*, and F_{B^*} will not involve x_r. Thus, $o(F) = o(F_{B^*}) \leqslant n-1$, a contradiction.

We can now proceed with the proof of Lemma B. Multiplying through by a constant, we may suppose that F has integer coefficients, then by Lemma 7, $\rho(B) = \max\{\|R\|, R$ a positive rake for $F_B\}$ exists, as does $\rho = \max\{\rho(B), B$ admissible$\}$. Hence there exist admissible B with $\rho(B) = \rho$, and associated with these

B are rakes R for which $\| R \| = \rho(B) = \rho$. Each of these R's is maximal for F_B, and so by Lemma 6 each determines a batching and hence a partition, $n_0 + n_1 + \ldots + n_{d-1} = n$. Thus we can choose a B and an R such that

(i) $\| R \|$ is maximal, (i.e. $\| R \| = \rho$) (5)

and

(ii) subject to (i), $n_0, n_1, \ldots, n_{d-1}$ are successively as small as possible. (6)

For simplicity of notation, suppose that $B = I$, then $F_B = F$.

The form F_R—applying Lemma 6—gives d forms ${}^0F_R{}^*$, ${}^1F_R{}^*, \ldots, {}^{d-1}F_R{}^*$ over k_q. Furthermore

$$ {}^jF_R{}^* = {}^jG^*(L_1^{(j)}, \ldots, L_{m_j}^{(j)}), $$

where $o({}^jF_R{}^*) = o({}^jG^*) = m_j$, and the L's are linear forms in $x_1, x_2, \ldots, x_n$, say $L_i^{(j)} = \sum_{r=1}^{n} L_{ir}^{(j)} x_r$ with each $L_{ir}^{(j)}$ in k. (Actually $L_{ir}^{(j)} = 0$ if x_r belongs to $\mathfrak{B}_k$ with $k > j$.) For each j there is a submatrix $\| \mathfrak{L}(j) \|$ whose rows correspond to the linear forms $L_i^{(j)}$ and whose columns correspond to the variables x_r which are in $\mathfrak{B}_j$; we assert that

$$ \text{rank } \| L(j) \| = n_j \text{ for all } j = 0, \ldots, d-1. \tag{7} $$

This is the crux of the argument. Once (7) has been proved it follows *a fortiori* that $m_j \geqslant n_j$; so

$$ \sum_{j=0}^{d-1} o({}^jF_R{}^*) = \sum_{j=0}^{d-1} m_j \geqslant \sum_{j=0}^{d-1} n_j = n; $$

hence for at least one j, $o({}^jF_R{}^*) \geqslant n/d$, that is, $o({}^jF_R{}^*) \geqslant u$. But the form $\pi^{d\|R\|+j}\,({}^jF_R)$ may be obtained from F by a non-singular integral transformation T, and we have (2) and (3) with

$$ F_T = \pi^{d\|R\|+j}\,({}^jF_R),\ H^* = {}^jF_R{}^*,\ \| F_T \| = d\,\| R \| + j. $$

Suppose on the contrary that (7) is false, and let J be such that

$$ \text{rank } \| \mathfrak{L}(J) \| \leqslant n_J - 1. $$

Then the columns of $\mathfrak{L}(J)$ are linearly dependent, so that for the $x_r \in \mathfrak{B}_J$ there are λ_r from k, not all zero, with

$$\sum_{r, x_r \in \mathfrak{B}_J} L_{ir}^{(J)} \lambda_r = 0 \text{ for } i = 1, \dots, m_J.$$

Set $\lambda_r = 0$ for $x_r \notin \mathfrak{B}_J$. Then there exist λ_r, not all zero, such that

$$\sum_{r=1}^{n} L_{ir}^{(J)} \lambda_r = 0 \text{ for } i = 1, 2, 3, \dots, m_J. \tag{8}$$

From among the x_σ for which $\lambda_\sigma \neq 0$, choose one having γ_σ maximal. The relation (8) is homogeneous, so we may assume, for simplicity, that $\lambda_\sigma = 1$; and by permuting the variables, we may suppose that x_σ is x_n. Then

$$L_i^{(J)} = \sum_{r=1}^{n} L_{ir}^{(J)} x_r = \sum_{r=1}^{n-1} L_{ir}^{(J)} x_r - \sum_{r=1}^{n-1} \lambda_r L_{ir}^{(J)} x_n = \sum_{r=1}^{n-1} L_{ir}^{(J)} (x_r - \lambda_r x_n),$$

so

$${}^J F_R^*(x_1, \dots, x_n) \equiv {}^J F_R^*(x_1 - \lambda_1 x_n, \dots, x_{n-1} - \lambda_{n-1} x_n, 0). \tag{9}$$

Pick $\mu_1, \dots, \mu_{n-1}$ in $\mathfrak{o}$ so that $\mu_r^* = \lambda_r$, and so that $\mu_r = 0$ whenever $\lambda_r = 0$, and define C by

$$F_C(x_1, \dots, x_n) = F(x_1 + \mu_1 \pi^{\gamma_n - \gamma_1} x_n, \dots, x_{n-1} + \mu_{n-1} \pi^{\gamma_n - \gamma_{n-1}} x_n, x_n);$$

then by the maximal property of γ_n, $\gamma_n - \gamma_r \geqslant 0$ unless $\mu_r = 0$, so C is admissible. Further,

$$F_{CR}(x_1, \dots, x_n) = F_R(x_1 + \mu_1 x_n, \dots, x_{n-1} + \mu_{n-1} x_n, x_n). \tag{10}$$

As F_R has integer coefficients, the same is true for F_{CR}; so by Lemma 4 there is a positive rake R' which is maximal for F_{CR}. R' must be the identity—indeed if it were not, RR' would be a rake for F_C with $\|RR'\| > \|R\|$, which would contradict the maximality condition (5). Therefore R is a maximal rake for F_C.

Accordingly, we may apply the process of Lemma 6 to F_C, and thus assign the variables $x_1, \dots, x_n$ to batches $\mathfrak{B}_0', \dots \mathfrak{B}'_{d-1}$, and obtain forms ${}^j F_{CR}$ so that every ${}^j F_{CR}$ has integer coefficients, and F_{CR}^* involves all the variables of $\mathfrak{B}'_j$, and no variables from any later batch.

A monomial of F_{CR} will either be one of F_R, or may be obtained from a monomial of F_R by replacing some x_r's of $\mathfrak{B}_J$ by $\mu_r x_n$. Since $\mu_r = 0$ if $x_r \notin \mathfrak{B}_J$, it follows immediately from the construction in Lemma 6 that $\mathfrak{B}_j = \mathfrak{B}'_j$ for $j = 0, \ldots, J-1$; and so by (10),

$$ {}^J F_{CR}(x_1, \ldots, x_n) = {}^J F_R(x_1 + \mu_1 x_n, \ldots, x_{n-1} + \mu_{n-1} x_n, x_n). $$

No variables from batches later than $\mathfrak{B}_J$ can be introduced into ${}^J F_{CR}{}^*$, so $\mathfrak{B}'_J \subseteq \mathfrak{B}_J$. On the other hand, by (9),

$$ \begin{aligned} {}^J F_{CR}{}^* (x_1, \ldots, x_n) &= {}^J F_R{}^*(x_1 + \lambda_1 x_n, \ldots, x_{n-1} + \lambda_{n-1} x_n, x_n) \\ &= {}^J F_R{}^*(x_1, \ldots, x_{n-1}, 0). \end{aligned} $$

Thus ${}^J F_{CR}{}^*$ does not involve x_n, so x_n is in $\mathfrak{B}_J$ but not in $\mathfrak{B}'_J$; we have thus shown that $n_j = n'_j$ for $j = 0, \ldots, J-1$,

$$ n'_J \leqslant n_J - 1, $$

which contradicts the minimality condition (6). This proves (7), and hence Lemma B.

Remark 3. The method of proving Lemma B has essentially been the reduction of F to a normalized form. For future use, in § 6, we will say that an integral form F is a *normalized form* if (i) the only positive rake for F is the identity, so that Lemma 6 is applicable, and we can define 'associated forms' ${}^0F^*, \ldots, {}^{d-1}F^*$, and (ii) the matrices $\|\mathfrak{L}(j)\|$ defined as in the proof of Lemma B each has rank n_j.

If we further say that a form is *degenerate* if its order is less than its number of variables, we obtain as a corollary of our proof the concise statement :

" *A non-degenerate $\mathfrak{p}$-adic form is equivalent to a multiple of a normalized form.* "

Remark 4. From the point of view of algebraic geometry, Lemma B gives information as to what happens to a variety over a $\mathfrak{p}$-adic field when $\mathfrak{p}$ is specialized to zero. A theory of such specializations has been given by Shimura [19]—however, Shimura deals only with absolutely irreducible varieties which specialize into

absolutely irreducible varieties; from our point of view, this is hardly helpful, as our major concern is to find out when a variety specializes into an absolutely irreducible one.

As a curiosity, we deduce

COROLLARY 5. *Let $F(x)$ be a form of degree d over K, with $o(F) = n \geqslant d+1$, and suppose that d is a prime, and that q, the order of the residue class field, is large enough compared to d and is not a power of d. Then F is 'almost universal' in the sense that given any a of K there exists a rational integer γ and a vector $\mathfrak{x}$ of K^n such that*

$$F(\mathfrak{x}) = \pi^{\gamma}\, a.$$

PROOF. We may suppose that F has integral coefficients, and involves just $d+1$ variables. By Lemma B, we may make a linear transformation so that $F_T(\mathfrak{x}) = \pi^{|F_T|}\, G(\mathfrak{x})$, with $o(G^*) \geqslant 2$. Since d is prime, G^* can only be a power of another form when it is a power of a linear form; and this is impossible, since a power of a linear form has order one. Hence, by Corollary 3, G^* is universal. The result now follows easily, on the lines of the proof of Corollary 4.

5. For the proof of our theorem on quintic forms two more lemmas are needed.

LEMMA 8. *Let $F^*(x)$ be a form of degree d in n variables over k_q, and suppose that $o(F^*) = n \geqslant d+1$. Then $d \neq 1$; if $d = 2$, F^* is absolutely irreducible; if $d = 3$, either F^* is absolutely irreducible, or else F^* has a simple linear factor; if $d = 5$ there is the additional possibility that $F^* = Q^*C^*$, where Q^* is quadratic, C^* is cubic, and one of Q^* and C^* is absolutely irreducible.*

PROOF. d is not 1, for a linear form has order 1.

If $d = 2$, either F^* is reducible in k_q, in which case it is the product of two linear forms, and so has order at most 2; or it is reducible in an extension of k_q, when it is the product of two conjugate linear forms, and hence has order 2; or else, as is the case, it is absolutely irreducible.

If $d = 3$, and F^* is not absolutely irreducible, either F^* is reducible in k_q, in which case it certainly has a simple linear factor unless it is a cube of a linear form—and so would have order 1; or else F^* is irreducible in k_q but reducible in an extension; in this case the extension can only be a cubic one and F^* is the product of three conjugate linear forms, and so $o(F^*) \leqslant 3$, contrary to the hypothesis.

If $d = 5$, F^* again may be either absolutely irreducible, or reducible in k_q, or irreducible in k_q but reducible in an extension of k_q. This last case may be dealt with as before—the extension can only be a quintic one, F^* is the product of five conjugate linear forms, and so $o(F^*) \leqslant 5$. If F^* is reducible in k_q, either F^* has a simple linear factor; or F^* is the fifth power of a linear form, in which case $o(F^*) = 1$; or else F^* is of the form Q^*C^*, where Q^* and C^* do not have a common factor, and neither Q^* nor C^* has a simple linear factor—by the case $d = 2$, either $o(Q^*) \leqslant 2$ or Q^* is absolutely irreducible and by the case $d = 3$ either $o(C^*) \leqslant 3$ or C^* is absolutely irreducible. So either $o(F^*) \leqslant 5$ or at least one of Q^* and C^* is absolutely irreducible.

LEMMA 9. *Let F^* be as in Lemma 8, and suppose that q is large enough compared to d. Then, if $d = 2$, 3 or 5, F^* has a non-singular zero.*

PROOF. If F^* has a simple linear factor, the result is obvious. If F^* is absolutely irreducible, the result follows from Corollary 2. There remains the case $F^* = Q^* C^*$, where one of Q^*, C^* is absolutely irreducible and Q^* does not divide C^*.

There are two possibilities. On the one hand, if Q^* is absolutely irreducible, it has more than $\frac{1}{2} q^{n-1}$ non-singular zeros; unless these are all zeros of C^*, one of them is a non-singular zero of F^*. On the other hand, if C^* is absolutely irreducible, it has more than $\frac{1}{2} q^{n-1}$ non-singular zeros; and unless these are all zeros of Q^*, one of them is a non-singular zero of F^*. Either way, F^* certainly has a non-singular zero unless Q^* and C^* have more than $\frac{1}{2} q^{n-1}$

zeros in common, all of which would be singular zeros of F^*; but then Lemma 2 implies that F^* has a squared factor, which is not so. This proves the lemma.

THEOREM 2. *Let $F(x)$ be a form over the $\mathfrak{p}$-adic field K, of degree d in d^2+1 variables. Then, for $d = 1, 2, 3,$ or 5, if the residue class field is sufficiently large $F(x)$ has a non-trivial zero.*

PROOF. Suppose that $F(\mathfrak{x})$ has no non-trivial zero—then *a fortiori* $o(F) = d^2+1$, so Lemma B is applicable, and after a non-singular transformation we may write $F(\mathfrak{x}) = \pi^{|FT|}\, G(\mathfrak{y})$, where $o(G^*) \geqslant d+1$. By Lemma 9, G^* has a non-singular zero, and by Lemma A, this induces a zero of F.

REMARKS. Our proof works essentially because when d is a small prime the forms whose order exceeds d only escape being absolutely irreducible in one of several rather innocuous ways. When d is composite, e.g. $d = 4$, a form such as $Q_1^2 + Q_2^2$ may be irreducible but not absolutely irreducible—Corollary 2 then gives us no non-singular zeros, and the proof collapses at Lemma 9. We might expect to do something for prime degrees greater than five—but the difficulties we encounter in the case $d = 4$ continue; for instance, a form of degree 7 may be a cube of a linear form multiplied by one of our unpleasant quartic forms about which Corollary 2 gives us no useful information.

The hypothesis that q is large enough is of course unnecessary in the cubic case, by Remark 1 after Corollary 2; this has been shown already in [10] and [20].

6. Finally, we make a few observations about what happens 'in general', and about the shape of those forms which involve a fairly large number of variables, but nevertheless have no zero.

Let k be a finite field, and let k' be a proper extension of k of degree n; let F be a form with coefficients in k', and let $F_1, \ldots, F_n$ be a complete set of conjugates of F over k. Then $N_{k'/k}\, F = \prod_{i=1}^{n} F_i$ is

a form with coeffiicents in k, we call it the *norm* of F. Examples of norms aro (i) proper powers of a form—when the form F happens to have its coefficients in k—(ii) irreducible binary forms of degree at least two with leading coefficient 1—if $g(x,y)$ is such a form, there is a normal extension k' of k in which g splits into linear factors, $g(x,y) = (x - \theta_1 y)\dots(x - \theta_n y)$, and then $g(x,y) = N_{k'/k}(x - \theta y)$ and—(iii) if $g(x_1, x_2, \dots, x_m)$ is a norm form of degree d and $h_1, h_2, \dots, h_m$ are forms of degree e over k, then $G = g(h_1, h_2, \dots, h_m)$ is a norm of degree de; e.g. $Q_1^2 + Q_2^2$, where Q_1 and Q_2 are quadratic forms and (-1) is a quadratic non-residue. Note that if α is in k, there exists α' of k' such that $\alpha = N_{k'/k}\,\alpha'$; hence a constant multiple of a norm is a norm.

Wo say that a form over a finite k is *bad* if it is a constant times a product of norms; a *good* form is one which is not bad. It follows that a good form has at least one simple absolutely irreducible factor—so combining Lemma 2 and Corollary 2 as in the proof of Lemma 9, we see that

If F^* is a good form of degree d in n variables over k_q, and $q \geqslant C(n,d)$, then F^* has a non-singular zero in k_q.

Referring to Lemma B, and Remark 3 at the end of its proof we deduce:

If H is a normalized $\mathfrak{p}$-adic form, then unless all the associated forms ${}^0H^*, \dots {}^{d-1}H^*$ are bad H has a non-trivial zero.

'Almost all' forms F^* of degree d in n variables, where $n \geqslant 3$, will be absolutely irreducible, in the sense that if F^* is not absolutely irreducible its coefficients will satisfy at least one of a finite set of algebraic identities; *a fortiori*, over a finite field.

Almost all forms in at least three variables are good.

Hence, with a reasonable interpretation, we may assert

If $q \geqslant E(n, d)$, then almost all normalized $\mathfrak{p}$-adic forms H of degree d in $(2d + 1)$ variables have a non-trivial zero.

Indeed, if H has no zero then $o(H) = 2d + 1$ and hence one of the associated forms ${}^0H^*, \ldots, {}^{d-1}H^*$ has $o({}^jH^*) \geqslant 3$; then ${}^jH^*$ will almost always be a good form, so that ${}^jH^*$ has a non-singular zero which induces a zero of H.

Lemma 8 may be rephrased as

If $d = 2$, 3, or 5, *every form* F^* *with* $o(F^*) \geqslant d + 1$ *is good.*

Furthermore

A good quadratic or cubic form always has a non-singular zero; a good quintic form over a sufficiently large field always has a non-singular zero,

and of course that is all that is needed for the proof of Theorem 2.

Referring to the proof of Lemma 8, we may make a more detailed assertion.

A bad quadratic or cubic form can only be a norm; a bad quintic form may be either a norm or else the product of a quadratic norm by a cubic norm.

To complete the picture we make several observations regarding $\mathfrak{p}$-adic forms without non-trivial zeros. Every quadratic form over a $\mathfrak{p}$-adic field K can be diagonalized. If this quadratic form contains 4 variables and has only the trivial zero, it is equivalent to

$$F = (ax^2 + by^2) + \pi(cz^2 + dw^2),$$

where a, b, c and d are units and $(-ab)^*$ and $(-cd)^*$ are quadratic non-residues. It follows that $ax^2 + by^2$ and $cz^2 + dw^2$ are norms of quadratic unramified extensions of K (hence ${}^0F^*$ and ${}^1F^*$ are norms), and that F is the norm of a quaternion algebra over K.

Every quadratic form in 4 *variables over a* $\mathfrak{p}$*-adic field* K*, having only the trivial zero in* K^4*, is the norm of a quaternion algebra over* K.

For cubic and quintic forms one does not obtain such a nice normalization. Let G be a normalized $\mathfrak{p}$-adic form of degree d in d^2 variables, having only the trivial zero; then ${}^0G^*, {}^1G^*, \ldots, {}^{d-1}G^*$

have only singular zeros. If $d = 3$ or if $d = 5$ and the residue class field is sufficiently large, we have

$$n_j \leqslant m_j = o({}^jG^*) \leqslant d, \quad j = 0, 1, \ldots, d-1.$$

But $d^2 \leqslant \sum_{j=0}^{d-1} n_j \leqslant \sum_{j=0}^{d-1} m_j \leqslant d^2$, hence for each j, $n_j = m_j = d$, and each ${}^jG^*$ is bad. It follows that there exists a transformation E, such that $F = G_E$ is normalized, and each ${}^jF^*$ is bad and involves only the d variables in $\mathfrak{B}_j$.

Every cubic form in 9 variables over a $\mathfrak{p}$-adic field K and having only the trivial zero in K, is equivalent to a form F, where the ${}^jF^*$ $(j = 0, 1, 2)$ are each the norm function of the cubic extension of the residue class field K^* and involve distinct sets of 3 variables. If $F = F_0 + \pi F_1 + \pi^2 F_2 + \ldots$, where the F_j have units as coefficients, then the F_j need not be norms over K, for example:

$$\sum_{j=0}^{2} 7^j(x_{3j}^3 + 2x_{3j+1}^3 + 4x_{3j+2}^3 - 6x_{3j}x_{3j+1}x_{3j+2}) + 7^9 x_0^3 + 7x_1x_4x_7.$$

Furthermore, so long as K^* is sufficiently large, every quintic form in 25 variables over the $\mathfrak{p}$-adic field K, having only the trivial zero in K, is equivalent to a form F, where the ${}^jF^*$ $(j = 0, 1, \ldots, 4)$ are each the norm function of the extension of K^* of degree 5, and involve distinct sets of 5 variables.

The above result for cubics follows from what has already been stated, but in the quintic case a bad quintic need not be a norm but can be the product of a quadratic norm and a cubic norm. Whenever some ${}^jF^*$ is such a product, one can find an admissible transformation which leaves the variables not in $\mathfrak{B}_j$ fixed and such that the quadratic and cubic factors involve distinct variables. Henceforth we suppose that F is so arranged. Since the ${}^jF^*$ involves distinct variables, the residue forms of ${}^jF$ are a cyclic permutation of ${}^0F^*, {}^1F^*, \ldots, {}^4F^*$; thus if some ${}^jF^*$ is reducible, we may assume that ${}^0F^*$ is reducible. Suppose ${}^0F^* = Q^*(x_1, x_2)C^*(x_3, x_4, x_5)$, where Q^* is a quadratic norm and C^* is a cubic norm. Let T be the transformation: $x_i = \pi x_i$ if $i = 1,2$; $x_i = x_i$ if $i \geqslant 3$. Set $H = \pi^{-1}F_T$, then ${}^0H^* = {}^1F^* + W^*$,

where each term of W^* involves at least one of x_3, x_4, or x_5. If ${}^0H^*$ does not have a non-singular zero (a possibility, since W^* can be 0) $o({}^0H^*) = 5$ and we can make a linear transformation S, which leaves the variables not in $\mathfrak{B}_1$ fixed and such that ${}^0H_S{}^* = {}^1F^*$. But now

$$ {}^1H_S{}^* = {}^2F^* + Q^* C^* + U^*, $$

where each term of U^* involves a variable of $\mathfrak{B}_0$ and no variable of $\mathfrak{B}_1$ also no term of U^* contains x_1 to a power exceeding 1. Since x_1^5 does not appear in ${}^1H_S{}^*$, but x_1^2 does, ${}^1H_S{}^*$ cannot be a norm. If ${}^1H_S{}^*$ is reducible, say

$$ {}^1H_S{}^* = Q_1{}^*C_1{}^* = {}^2F^* + Q^* C^* + U^*, $$

where $Q_1{}^*$ and $C_1{}^*$ are quadratic and cubic forms respectively, it follows that $Q_1{}^*(x_1, \dots, x_5, 0, \dots, 0)\ C_1{}^*(x_1, \dots, x_5, 0, \dots, 0) = Q^* C^* + u^*(x_1, \dots, x_5, 0, \dots, 0)$. Thus $C_1{}^* = C^* + N$, where each term of N contains variables of $\mathfrak{B}_2$. If $N \neq 0$, then x_1^2 must appear as a factor of some term of U^*; while if $N = 0$, then ${}^2F^* = 0$; in either case we have a contradiction. Hence either ${}^0H^*$ or ${}^1H^*$ has a non-singular zero and consequently F has a non-trivial zero in K.

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University of Notre Dame
Notre Dame, Indiana U.S.A.

POINTWISE BOUNDS FOR THE SOLUTIONS OF CERTAIN BOUNDARY VALUE PROBLEMS

By R. MANOHAR

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INTRODUCTION : Using Schwarz inequality, upper and lower bounds for the solutions of the first biharmonic boundary value problem, the Dirichlet problem, Neumann and mixed type of problems have been obtained by Diaz [2], Diaz and Greenberg [3], Cooperman [1] and others. Their method is analytic and has been applied to each problem separately to derive bounds for the solutions and their derivatives. On the other hand a geometric method, which seems to be more general, has been used by Synge [7] and Maple [5] and is known as the method of hypercircle in function-space. The purpose of this study is to find a relation between the two methods, and to see whether the two methods yield results which are the same, as far as the application is concerned, or not. The bounds obtained by the hypercircle method were essentially better than those obtained by Diaz and Greenberg [3]. Most of the terms appearing there were similar to those obtained by the hypercircle method. This fact led us to believe that the inequalities obtained by Diaz could be modified to get the same results as given by the hypercircle method.

In §1 the inequalities obtained by Diaz and Greenberg [3] for the first biharmonic problem have been modified. Most of the notations and the results of his paper have been used. The method of hypercircle has been applied to the same problem in §2 and it has been shown that all the results of §1 follow directly by properly choosing the vector G in function-space. The same procedure can be applied to the problems of Dirichlet and Neumann and the results hold good in n-dimensions as well, although for the sake of simplicity we restrict ourselves to 2-dimensions only.

1. Let the biharmonic differential equation be

PROBLEM I:

$$\Delta\Delta\, w = p(x, y) \quad \textit{in the region } A, \tag{1.1}$$

and

$$w = 0 : \frac{\partial w}{\partial n} = 0 \textit{ on } B. \tag{1.2}$$

(Without loss of generality, the boundary conditions can be taken to be homogeneous, because a problem with non-homogeneous boundary conditions can be reduced to a problem of the type stated above.)

Here $\Delta \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$, A is a plane domain with boundary B. The function $p(x, y)$ is given and under suitable restrictions on p and A the Problem I possesses a unique solution. It is proposed to find upper and lower bounds for the solution $w(x_0, y_0)$ at (x_0, y_0) a given interior point of A. Following Diaz† we define

$$\left.\begin{aligned} &\iint_A \Delta\phi\, \Delta\psi\; dxdy = H(\phi, \psi) \\ \text{and} \quad &\iint_A (\Delta\phi)^2\, dxdy = H(\phi) \geqslant 0. \end{aligned}\right\} \tag{1.3}$$

Let w' be a function satisfying the differential equation (1.1) and w'' a function satisfying the boundary conditions (1.2) such that

$$\Delta\Delta\, w' = p(x, y) \text{ in } A \text{ and} \tag{1.4}$$

$$w'' = w = 0 : \frac{\partial w''}{\partial n} = \frac{\partial w}{\partial n} = 0 \text{ on } B. \tag{1.5}$$

Then from Schwarz inequality, it follows that

$$\left.\begin{aligned} H(w'' - w) \\ H(w' - w) \end{aligned}\right\} \leqslant H(w'' - w'). \tag{1.6}$$

† Most of the notations used by Diaz and Greenberg have been retained except that we use $u \equiv w''$, $v \equiv w'$, $\bar{v} \equiv R'$, $\bar{u} \equiv R''$ and $\bar{w} \equiv R$.

The equation (1.6) is the same as equation (6) given by Diaz and Greenberg [3]. Further let R be a function satisfying

PROBLEM II:

$$\Delta\Delta\, R = 0 \textit{ in } A \tag{1.7}$$

and

$$R = -\, r^2 \log r : \frac{\partial R}{\partial n} = -\, \frac{\partial r^2 \log r}{\partial n} \textit{ on } B. \tag{1.8}$$

Here R is a regular function associated with the Green's function g with pole at (x_0, y_0) which is defined by

$$g(x, y : x_0, y_0) = r^2 \log r + R(x, y). \tag{1.9}$$

Analogous to w' and w'' associated with the Problem I above, we may consider functions R' and R'' such that

$$\Delta\Delta\, R' = 0 \text{ in } A \tag{1.10}$$

and

$$R'' = R = -\, r^2 \log r : \frac{\partial R''}{\partial n} = \frac{\partial R}{\partial n} = -\, \frac{\partial (r^2 \log r)}{\partial n} \text{ on } B. \tag{1.11}$$

Then from Schwarz inequality, it follows that

$$\left.\begin{array}{l} H(R'' - R) \\ H(R' - R) \end{array}\right\} \leqslant H(R'' - R'). \tag{1.12}$$

If we now put

$$q = w + \epsilon R,\ q' = w' + \epsilon R', \text{ and } q'' = w'' + \epsilon R'', \tag{1.13}$$

where ϵ is any parameter (real), we then have

PROBLEM III:

$$\Delta\Delta\, q = p(x, y) \textit{ in } A \tag{1.14}$$

and

$$q = -\, \epsilon\, r^2 \log r : \frac{\partial q}{\partial n} = -\, \epsilon\, \frac{\partial}{\partial n}\, (r^2 \log r) \textit{ on } B. \tag{1.15}$$

Also q' satisfies the differential equation (1.14) and q'' the boundary conditions (1.15) such that

$$\Delta\Delta\, q' = p(x, y) \text{ in } A \tag{1.16}$$

and

$$q'' = -\epsilon\, r^2 \log r : \frac{\partial q''}{\partial n} = -\epsilon \frac{\partial}{\partial n}(r^2 \log r) \text{ on } B. \tag{1.17}$$

Analogus to (1.6) we can write for Problem III the inequalitios

$$\left.\begin{matrix} H(q'' - q) \\ H(q' - q) \end{matrix}\right\} \leqslant H(q'' - q'). \tag{1.18}$$

If we substitute the values of q, q' and q'' from (1.13) in (1.18) we get

$$l + m\epsilon + n\epsilon^2 \leqslant \alpha + \beta\epsilon + \gamma\epsilon^2 \leqslant L + M\epsilon + N\epsilon^2, \tag{1.19}$$

where

$$\left.\begin{aligned} & l = H(w'', w') - \tfrac{1}{2} H(w''), m = \iint_A p\, R''\, dx\, dy - H(w'', R''), \\ & \qquad n = -\tfrac{1}{2} H(R''), \\ & L = \tfrac{1}{2} H(w'), M = H(w', R') - \int_B \left(\Delta w' \frac{\partial R}{\partial n} - R \frac{\partial \Delta w'}{\partial n}\right) ds, \\ & \qquad N = \tfrac{1}{2} H(R') - H(R''\ R'), \\ & \alpha = \tfrac{1}{2} H(w), \beta = -\int_B \left(\Delta w \frac{\partial R}{\partial n} - R \frac{\partial \Delta w}{\partial n}\right) ds \\ & \qquad = (8\pi w - \sigma) \text{ (see below)}, \\ & \qquad \gamma = -\tfrac{1}{2} H(R). \end{aligned}\right\} \tag{1.20}$$

From (1.6) and (1.12) it follows that

$$l \leqslant L \text{ and } n \leqslant N. \tag{1.20 a}$$

Further using the relation

$$8\pi w(x_0, y_0) = \iint_A r^2 \log r \,.\, p\, dx\, dy + $$

$$+ \int_B \left(\Delta w \frac{\partial r^2 \log r}{\partial n} - r^2 \log r \frac{\partial \Delta w}{\partial n}\right) ds \tag{1.20 b}$$

we find that

$$\left.\begin{aligned} \beta &= 8\pi\, w(x_0, y_0) - \sigma \\ \text{with} \qquad \sigma &= \iint_A r^2 \log r \,.\, p \, dx \, dy. \end{aligned}\right\} \qquad (1.21)$$

The inequalities (1.19) hold good whatever the value of ϵ may be. We therefore write $-\epsilon$ for ϵ and subtract the result from (1.19) and get

$$\begin{aligned} &[-(L-l) - (M+m)\epsilon - (N-n)\epsilon^2] \\ &\leqslant -2\beta\epsilon \\ &[\leqslant (L-l) - (M+m)\epsilon + (N-n)\epsilon^2] \end{aligned}$$

which may be written as

$$(L-l) \pm (M+m-2\beta)\epsilon + (N-n)\epsilon^2 \geqslant 0^{\ddagger}. \qquad (1.22)$$

Since this holds good for any arbitrary value of ϵ, it follows that the discriminant must never be positive, hence

$$4(L-l)(N-n) \geqslant (M+m-2\beta)^2$$

or

$$\begin{aligned} \tfrac{1}{2}(M+m) - \sqrt{\{(L-l)(N-n)\}} \leqslant \beta \\ &\leqslant \tfrac{1}{2}(M+m) + \\ &+ \sqrt{\{(L-l)(N-n)\}}. \end{aligned} \qquad (1.23)$$

Using (1.21) we may put this in the form

$$\begin{aligned} \tfrac{1}{2}(M+m) + \sigma - \sqrt{\{(L-l)(N-n)\}} \leqslant 8\pi\, w(x_0, y_0) \\ &\leqslant \tfrac{1}{2}(M+m) + \sigma + \\ &+ \sqrt{\{(L-l)(N-n)\}}. \end{aligned} \qquad (1.24)$$

Comparing (1.20) with the results given by Diaz and Greenberg [3] we find that

$$(L-l) = a/2,\ (N-n) = c/2,\ M = b' - \sigma,\ m = b - \sigma.$$

Hence (1.24) can be written as

‡ The author is thankful to Dr. K. Matschat for drawing his attention to the above inequality. Matschat [6] obtains it by following the technique of Funk and Berger [4].

$$\tfrac{1}{2}(b+b') - \tfrac{1}{2}\sqrt{(ac)} \leqslant 8\pi w(x_0, y_0) \leqslant \tfrac{1}{2}(b+b') + \tfrac{1}{2}\sqrt{(ac)}. \quad (1.25)$$

The difference between the upper and lower bounds is

$$D_1 = \sqrt{(ac)}. \quad (1.26)$$

The results obtained by Diaz and Greenberg are

$$[8\pi w(x_0, y_0) - b]^2 \leqslant ac \quad \text{and} \quad [8\pi w(x_0, y_0) - b']^2 \leqslant ac.$$

The best approximation, if $b > b'$, is

$$b - \sqrt{(ac)} \leqslant 8\pi w(x_0, y_0) \leqslant b' + \sqrt{(ac)}; \quad (1.27)$$

if $b' > b$ then we interchange b and b' to get the best approximation. In any case, the difference between the upper and lower bounds is

$$-|b' - b| + 2\sqrt{(ac)} = D_2. \quad (1.28)$$

Consider now the inequality (1.19) in the form

$$(L - l) + (M - m)\epsilon + (N - n)\epsilon^2 \geqslant 0.$$

Since this holds for any arbitrary value of ϵ, we have

$$4(L - l)(N - n) \geqslant (M - m)^2,$$

which gives $(ac) \geqslant (b' - b)^2$.

It is therefore obvious that from (1.26) and (1.28)

$$D_2 \geqslant \sqrt{(ac)} = D_1.$$

The sign of equality holds only in a very special case when $b' = b$, hence the bounds given by (1.25) are better than those given by (1.27).

In practice, the bounds are improved by adding to w' a linear combination of functions w'_ρ and replacing w'' by a linear combination of functions w''_σ such that

$$\Delta\Delta w'_\rho = 0 \text{ in } A \text{ and } w''_\sigma = \frac{\partial w''_\sigma}{\partial n} = 0 \text{ on } B \quad (1.29)$$

for $\rho = 1, 2, \ldots, r$ and $\sigma = 1, 2, \ldots, s$.

These functions are orthonormalized so that

$$H(w'_\mu, w'_\rho) = \delta_{\mu\rho}, H(w''_\nu, w''_\sigma) = \delta_{\nu\sigma} \quad (1.30)$$

and obviously

$$H(w'_\rho, w''_\sigma) = 0.$$

Thus replacing w' and w'' by v' and v'' such that

$$v' = w' + \sum_{\rho=1}^{r} a'_\rho w'_\rho, \quad v'' = \sum_{\sigma=1}^{s} a''_\sigma w''_\sigma, \tag{1.31}$$

and R' and R'' by Q' and Q'' such that

$$Q' = \sum_{\rho=1}^{r} \bar{a}'_\rho w'_\rho, \; Q'' = R'' + \sum_{\sigma=1}^{s} \bar{a}'_\sigma w''_\sigma, \tag{1.32}$$

the coefficients a'_ρ, a''_σ, $\bar{a}'_\rho$ and $\bar{a}''_\sigma$ are determined to make a and c minimum. The values as obtained by Diaz and Greenberg are

$$a'_\rho = -H(w', w'_\rho), \quad a''_\sigma = H(w', w''_\sigma) \tag{1.33}$$

$$\bar{a}'_\rho = H(R'', w'_\rho), \qquad \bar{a}''_\sigma = -H(R'', w''_\sigma), \tag{1.34}$$

and the corresponding values of a, c, b and b' are then

$$\left.\begin{aligned}
a_{rs} &= H(w') - \sum_{\rho=1}^{r} (a'_\rho)^2 - \sum_{\sigma=1}^{s} (a''_\sigma)^2, \\
c_{rs} &= H(R'') - \sum_{\rho=1}^{r} (\bar{a}'_\rho)^2 - \sum_{\sigma=1}^{s} (\bar{a}''_\sigma)^2, \\
b_{rs} &= \iint_A pR''\,dx\,dy + \sum_{\sigma=1}^{s} a''_\sigma \bar{a}''_\sigma + \sigma, \\
b'_{rs} &= -H(w', R'') - \sum_{\rho=1}^{r} a'_\rho \bar{a}'_\rho + \sigma + \iint_A pR''\,dx\,dy.
\end{aligned}\right\} \tag{1.35}$$

The inequality (1.6) can first be written as

$$2H(w', w'') - H(w'') \leqslant H(w) \leqslant H(w') \tag{1.36}$$

or by substituting v' and v'' for w' and w'' and using the relations (1.29) to (1.35) we can write it as

$$\sum_{\sigma=1}^{s} (a''_\sigma)^2 \leqslant H(w) \leqslant H(w') - \sum_{\rho=1}^{r} (a'_\rho)^2. \tag{1.37}$$

Similarly the inequality (1.25) can be written as

$$\begin{aligned}
\tfrac{1}{2}(b_{rs} + b'_{rs}) - \tfrac{1}{2}\sqrt{(a_{rs}\, c_{rs})} &\leqslant 8\pi w(x_0, y_0) \\
&\leqslant \tfrac{1}{2}(b_{rs} + b'_{rs}) + \tfrac{1}{2}\sqrt{(a_{rs}\, c_{rs})}.
\end{aligned} \tag{1.38}$$

2. Let us now apply the method of hypercircle in function-space to solve the boundary value Problem I. Before we do this, we may quote some of the results arrived at by Synge [7]†.

For the solution we take the vector $\mathbf{S}$ in function-space. L', L'' = pair of intersecting orthogonal linear sub-spaces, L'' containing $\mathbf{O}$. $\mathbf{S}$ = their point of intersection. $\mathbf{S}'$, $\mathbf{S}''$ = any point on L', L'' respectively. L'_r, L''_s = linear r-space and linear s-space contained in L', L'' respectively, L''_s containing $\mathbf{O}$. Further $\mathbf{S}'_0$ = some point on L'_r. $\mathbf{I}'_\rho$ $(\rho = 1, 2, \ldots, r)$, $\mathbf{I}''_\sigma$ $(\sigma = 1, 2, \ldots, s)$ = orthogonal vectors lying in L'_r, L''_s respectively and satisfying the conditions

$$\mathbf{I}'_\mu \cdot \mathbf{I}'_\rho = \delta_{\mu\rho}, \mathbf{I}''_\nu \cdot \mathbf{I}''_\sigma = \delta_{\nu\sigma} \text{ and } \mathbf{I}'_\rho \cdot \mathbf{I}''_\sigma = 0. \tag{2.1}$$

Putting $\mathbf{V}' = \mathbf{S}'_0 + \sum_{\rho=1}^{r} a'_\rho \mathbf{I}'_\rho$, $\mathbf{V}'' = \sum_{\sigma=1}^{s} a''_\sigma \mathbf{I}''_\sigma$ with

$$a'_\rho = -(\mathbf{S}'_0 \cdot \mathbf{I}'_\rho) \text{ and } a''_\sigma = (\mathbf{S}'_0 \cdot \mathbf{I}''_\sigma), \tag{2.2}$$

the centre of the hypercircle $\mathbf{C}$ and the radius R are given by

$$\mathbf{C} = \tfrac{1}{2}(\mathbf{V}' + \mathbf{V}'') = \tfrac{1}{2}\left[\mathbf{S}'_0 + \sum_{\rho=1}^{r} a'_\rho \mathbf{I}'_\rho + \sum_{\sigma=1}^{s} a''_\sigma \mathbf{I}''_\sigma\right], \tag{2.3}$$

$$R = \tfrac{1}{2}\,|\mathbf{V}' - \mathbf{V}''| \text{ or } 4R^2 = \left[(\mathbf{S}'_0)^2 - \sum_{\rho=1}^{r}(a'_\rho)^2 - \sum_{\sigma=1}^{s}(a''_\sigma)^2\right], \tag{2.4}$$

and the following inequalities hold

$$2\mathbf{S}' \cdot \mathbf{S}'' - \mathbf{S}''^2 \leqslant \mathbf{S}^2 \leqslant \mathbf{S}'^2 \tag{2.5}$$

or

$$\sum_{\sigma=1}^{s}(a''_\sigma)^2 \leqslant \mathbf{S}^2 \leqslant (\mathbf{S}'_0)^2 - \sum_{\rho=1}^{r}(a'_\rho)^2. \tag{2.6}$$

Further the bounds on any vector $\mathbf{G}$ are given by

$$|\mathbf{S} \cdot \mathbf{G} - \mathbf{C} \cdot \mathbf{G}| \leqslant R\,|\mathbf{G}_0| \tag{2.7}$$

with

$$\mathbf{G}_0^2 = \mathbf{G}^2 - \sum_{\rho=1}^{r}(\mathbf{G} \cdot \mathbf{I}'_\rho)^2 - \sum_{\sigma=1}^{s}(\mathbf{G} \cdot \mathbf{I}''_\sigma)^2. \tag{2.8}$$

† pp. 118-120. The notation used here is the same as used by Synge.

The functions used below in defining the correspondence between the function-space and the physical-space are the same as have used used in § 1. We may thus write,

$$\mathbf{S} \longleftrightarrow w, \mathbf{S}' \equiv \mathbf{S}'_0 \longleftrightarrow w', \mathbf{S}'' \longleftrightarrow w'', \mathbf{I}'_\rho \longleftrightarrow w'_\rho, \mathbf{I}''_\sigma \longleftrightarrow w''_\sigma,$$

$$\mathbf{V}' \longleftrightarrow v' = w' + \sum_{\rho-1}^{r} a'_\rho\, w'_\rho \text{ and } \mathbf{V}'' \longleftrightarrow v'' = \sum_{\sigma=1}^{s} a''_\sigma\, w''_\sigma.$$

Defining the scalar product in F-space by $\mathbf{S}\cdot\mathbf{S}' = H(w, w')$ we get $\mathbf{S}^2 = H(w) \geqslant 0$ a positive-definite metric.

We can easily see that the relations (2.1), (2.2) are the same as (1.30) and (1.33) respectively and the inequalities (2.5), (2.6) are the same as (1.36) and (1.37). Further from (2.4) and (1.35) we have

$$4\,R^2 = (\mathbf{S}'_0)^2 - \sum_{\rho=1}^{r} (a'_\rho)^2 - \sum_{\sigma=1}^{s} (a''_\sigma) = a_{rs}. \qquad (2.9)$$

Let us now consider the vector G such that

$$G \longleftrightarrow (Q'' - Q') = R'' + \sum_{\sigma=1}^{s} \bar{a}''_\sigma\, w''_\sigma - \sum_{\rho=1}^{r} \bar{a}'_\rho\, w'_\rho.$$

(There is no such restriction as above in the choice of the vector $\mathbf{G}$, as far as the theory of hypercircle method is concerned, but in practice some such choice as above is inevitable.) Since $|\mathbf{G}_0| \leqslant |G|$ we choose $\bar{a}''_\sigma$ and $\bar{a}'_\rho$ such that the value of $\mathbf{G}$ is a minimum. We thus obtain the same values for $\bar{a}''_\sigma$ and $\bar{a}'_\rho$ as given by (1.34). With these values we have

$$\sum_{\rho=1}^{r} (\mathbf{G}\cdot\mathbf{I}'_\rho) = \sum_{\sigma=1}^{s} (\mathbf{G}\cdot\mathbf{I}''_\sigma) = 0 \text{ and } |\mathbf{G}_0| = |\mathbf{G}|.$$

The inequality (2.7) is then

$$|\mathbf{S}\cdot\mathbf{G} - \mathbf{C}\cdot\mathbf{G}| \leqslant \mathbf{R}\,|\mathbf{G}| \qquad (2.10)$$

and from (1.35)

$$\mathbf{G}^2 = H(R'') - \sum_{\sigma=1}^{s} (\bar{a}''_\sigma)^2 - \sum_{\rho=1}^{r} (\bar{a}'_\rho)^2 = c_{rs}. \qquad (2.11)$$

Now we shall find **S·G** and **C·G** and substitute in the inequality (2.10). We have

$$\mathbf{S}\cdot\mathbf{G} = H(w, R'') + \sum_{\sigma=1}^{s} \bar{a}''_\sigma a''_\sigma, \quad \mathbf{C}\cdot\mathbf{G} = \tfrac{1}{2} H(w', R'') +$$
$$+ \tfrac{1}{2}\sum_{\sigma=1}^{s} a''_\sigma \bar{a}''_\sigma + \tfrac{1}{2}\sum_{\rho=1}^{r} a'_\rho \bar{a}'_\rho.$$

Hence from (1.20a) and (1.35)

$$\mathbf{S}\cdot\mathbf{G} - \mathbf{C}\cdot\mathbf{G} = \iint_A p\, R''\, dx\, dy - \tfrac{1}{2} H(w', R'') +$$
$$+ \tfrac{1}{2}\sum_{\sigma=1}^{s} a''_\sigma \bar{a}''_\sigma - \tfrac{1}{2}\sum_{\rho=1}^{r} a'_\rho \bar{a}'_\rho - \beta$$
$$= \tfrac{1}{2}(b_{rs} + b'_{rs}) - \beta - \sigma$$
$$= \tfrac{1}{2}(b_{rs} + b'_{rs}) - 8\pi\, w(x_0, y_0). \tag{2.12}$$

The inequality (2.10) can thus be written with the help of (2.9), (2.11) and (2.12) as

$$\left|\tfrac{1}{2}(b_{rs} + b'_{rs}) - 8\pi\, w(x_0, y_0)\right| \leqslant \tfrac{1}{2}\sqrt{(a_{rs}\, c_{rs})}$$

which is the same as (1.38).

CONCLUSION. The bounds for the derivatives of the solutions can also be found by proceeding exactly as above by either method and by choosing the derivatives of the Green's function instead of R. The hypercircle method is obviously more general because the inequalities have been obtained once for all and the application to a particular problem only requires defining the scalar product in function-space. So far as the bounds at a point are concerned there seems to be very little choice left save using the singular part of the Green's function, as is done in case of analytic methods as well.

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Panjab Engineering College
Chandigarh

ON THE TRANSLATIVITY OF HAUSDORFF-AND SOME RELATED METHODS OF SUMMABILITY*

By M. R. PARAMESWARAN

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1. Introduction. The general problem of translativity for Hausdorff methods (H, μ_n) has been studied by Agnew [1] and recently by Kuttner [7] who has obtained (i) a set of necessary conditions, and (ii) a set of sufficient conditions, for translativity; Kuttner's conditions are all on the Hausdorff matrix and he makes no restrictions on the sequences involved. The question of translativity for quasi-Hausdorff methods (H^*, ν_n) has been treated by Ramanujan [17], but he restricts himself to bounded sequences only. A result due to Kuttner and given by Ramanujan [17] is that conservative quasi-Hausdorff methods are translative for bounded sequences. More recently, this result has been extended by Kuttner [10] to Hausdorff methods also.† In the present paper I study the problem of translativity for Hausdorff methods (H, μ_n), quasi-Hausdorff methods (H^*, ν_n) and a related class of methods (S^*, ω_n) which may be called the Ramanujan-Laurent-series methods and which seem to have been investigated for the first time by Ramanujan in [16]; the problem is considered for suitably restricted classes of sequences only, but with little or no conditions on the summability methods themselves. The results obtained include, in particular, a generalization of Kuttner's results quoted above. The similarity of the results (and of their proofs) for the methods (H, μ_n), (H^*, ν_n) and (S^*, ω_n) emphasize the close relationship between these classes of methods first pointed out by Ramanujan [16] and which has been investigated further by Kuttner [8], [9] in certain special

* Read at the Twentyfourth Conference of the Society, (Poona, December, 1958).

† I am thankful to Dr. B. Kuttner for this information and also for other remarks. The results presented here were obtained independently and are more general than his; the proof adopted here reflects the close relationship between the families (H, μ_n), (H^*, ν_n) and (S^*, ω_n). See also remark after Theorem 7.2.

cases. It is found that the (S^*, μ_n) methods behave, in certain respects, somewhat differently from the (H, μ_n) and (H^*, ν_n) methods (see Theorem 8 and the remark following it). An interesting application of the main theorems obtained in the present paper is to the summability of sequences of 0's and 1's (§ 6).

2. Some definitions. It will be convenient to begin by recalling certain definitions and known results. Consider any transformation of the sequence $\sigma = \{\sigma_n\}$ into the sequence $\tau = \{\tau_n\}$ by means of the equations

$$\tau_n = \sum_{k=0}^{\infty} a_{nk}\, \sigma_k \quad (n = 0, 1, \ldots,)^{\dagger}. \tag{1}$$

The transformation (1), and the matrix A belonging to it, are said to be *conservative* if $\sigma_n \to l_1$ implies $\tau_n \to l_2$ (as $n \to \infty$) and to be *regular*, if, in addition $l_1 = l_2$.

If $\sigma = \{\sigma_n\}$ be a given sequence, we can form from it two sequences $\overrightarrow{\sigma}$, $\overleftarrow{\sigma}$ called respectively the *right* and *left translates of* σ, in the follwing manner :

$$\overrightarrow{\sigma} = \{\overrightarrow{\sigma_n}\} = \{\sigma_{n+1}\}, \quad \overleftarrow{\sigma_n} = \{\overleftarrow{\sigma_n}\} = \{\sigma_{n-1}\},$$

where $\overleftarrow{\sigma_0} = \sigma_{-1}$ is an arbitrary fixed number.†† We shall denote by $\overrightarrow{\tau} = \{\overrightarrow{\tau_n}\}$ the transform of $\overrightarrow{\sigma}$ by, say, a transformation of the type (1); similarly we define the transform $\overleftarrow{\tau}$. (It is to be noted that neither $\overrightarrow{\tau}$ nor $\overleftarrow{\tau}$ need be a translate of τ; the arrow indicates a translate when used with the sequence denoted by σ only,

† All sequences and numbers considered in this paper may be real or complex. In all sequences $\{\sigma_n\}$, $\{\sigma_{n-1}\}$, etc. n ranges from 0 to ∞. All limits considered are supposed to be finite.

†† It has been the custom further to assume that $\sigma_{-1} = 0$. The value of σ_{-1} does not affect translativity (or absolute regularity) in the wide sense, of conservative methods, nor the translativity (or the absolute regularity) of multiplicative methods,

It may also be noted that Zeller [18] defines right translativity as the left translativity defined here, and vice versa.

throughout this paper.) We use also the symbol $\bar{\sigma}$ to denote, consistently, either of $\overrightarrow{\sigma}$, $\overleftarrow{\sigma}$.

Define the method A given by the transformation (1) to be

(1) absolutely regular to the left in the wide sense for the sequence σ if $\lim (\tau_n - \overleftarrow{\tau}_n) = l_1$ exists;

(2) absolutely regular to the right in the wide sense for the sequence σ if $\lim (\tau_n - \overrightarrow{\tau}_n) = l_2$ exists;

(3) absolutely regular to the left in the strict sense for the sequence σ if (1) holds with $l_1 = 0$;

(4) absolutely regular to the right in the strict sense for the sequence σ if (2) holds with $l_2 = 0$;

(5) almost absolutely regular for the sequence σ if it is absolutely regular to the left as well as to the right, in the wide sense, for σ; i.e. if both (1) and (2) hold;

(6) absolutely regular in the wide sense for the sequence σ if (1) and (2) hold with $l_1 = l_2$;

(7) absolutely regular in the strict sense for the sequence σ if (3) and (4) hold;‡

(8) translative to the left in the wide sense for the sequence σ if the existence of $\lim \tau_n = l$ implies that of $\lim \overleftarrow{\tau}_n = l_1$;

(9) translative to the left in the strict sense for the sequence σ if (8) holds with $l_1 = l$;

(10) translative to the right in the wide sense for the sequence σ if the existence of $\lim \tau_n = l$ implies that of $\lim \overrightarrow{\tau}_n = l_2$;

‡ The use of the phrase "absolutely regular" in the sense of (7) is due to Cooke [2]; it is used in quite a different sense in absolute summability. The terminology is in neither case very satisfactory.

(11) translative to the right in the strict sense for the sequence σ if (10) holds with $l_2 = l$;

(12) almost translative for the sequence σ if both (8) and (10) hold;

(13) translative in the wide sense for the sequence σ if (8) and (10) hold, with $l_1 = l_2$;

(14) translative in the strict sense for the sequence σ if (9) and (11) hold.

3. The methods (H, μ_n), (H^*, μ_n) and (S^*, μ_n). Corresponding to any sequence $\{\mu_n\}$ we may form the following transformations :

$$\tau_n = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} \mu_k \sigma_k \quad (n = 0, 1, \dots) \tag{2}$$

$$\tau^*_n = \sum_{k=n}^{\infty} \binom{k}{n} \Delta^{k-n} \mu_n \sigma_k \quad (n = 0, 1, \dots) \tag{3}$$

$$\lambda^*_n = \sum_{k=0}^{\infty} \binom{n+k}{k} \Delta^{k} \mu_n \sigma_k \quad (n = 0, 1, \dots), \tag{4}$$

where

$$\Delta \mu_n = \mu_n - \mu_{n+1}, \ \Delta^0 \mu_n = \mu_n, \ \Delta^k \mu_n = \Delta(\Delta^{k-1} \mu_n)$$

and the numbers τ^*_n, λ^*_n are assumed to exist. The transformations (2)-(4) are of the form (1), and their matrices are, respectively, the *Hausdorff* matrix (H, μ_n), the *quasi-Hausdorff* matrix (H^*, μ_n) and the *Ramanujan-Laurent* matrix (S^*, μ_n). We recall the necessary and sufficient conditions for these methods to be conservative : *(H, μ_n) is conservative if and only if $\{\mu_n\}$ is a sequence of moment constants ; each of (H^*, μ_n), (S^*, μ_n) is conservative if and only if $\{a, \mu_0, \mu_1, \dots, \dots\}$ is a sequence of moment constants, where a is any arbitrary number.* Here by the statement that $\{\mu_n\}$ is a sequence of moment constants we mean that there exists a function $\chi(t)$ of bounded variation in (0, 1) such that

$$\mu_n = \int_0^1 t^n \, d\chi(t) \quad (n \geqslant 0). \tag{5}$$

For every sequence $\{\mu_n\}$ of moment constants, it is known that

$$\lim \mu_n = \chi(1) - \chi(1-0)$$

exists. It is useful to note that an alteration of the value of the function $\chi(t)$ at $t = 1$ produces the same alteration in the value of each μ_n and therefore also in $\lim \mu_n$; on the other hand, the alteration of the value of $\chi(t)$ at $t = 0$ affects only μ_0, and conversely.

Let (H^*, μ_n), or (S^*, μ_n) be a conservative method. Then, for every number a, there is a function $\chi_a(t)$ of bounded variation in (0, 1) such that $\{a, \mu_0, \mu_1, ..., ...\}$ is the moment sequence generated by it. Since the value of $\chi_a(t)$ at $t = 0$ alone depends on the value of a, the quantities $\chi_a(1) - \chi_a(1-0)$ and $\chi_a(1) - \chi_a(+0)$ are independent of a. Now let $\chi(t)$ be the function of bounded variation generating the moment sequence

$$\{\chi_a(1) - \chi_a(+0), \mu_0, \mu_1, \mu_2, \dots\}.$$

Then $\chi(t)$ is continuous at $t = 0$; for,

$$\chi(+0) - \chi(0) = \int_0^1 d\chi - \int_{+0}^1 d\chi = \int_0^1 d\chi - \int_{+0}^1 d\chi_a = 0^\dagger.$$

4. Some properties of the methods $E(t)$, $T(t)$ **and** $L(t)$. In this section we collect together certain results on the behaviour of the *Euler-Knopp* method $E(t)$, the "*circle-*" or *Taylor* method $T(t)$ and the *Meyer-König* or *Laurent-series* method $L(t)$ (called S_{1-t} by Meyer-König and called S_t by Zeller [18]). These are methods (H, μ_n), (H^*, μ_{n+1}) and (S^*, μ_{n+1}) respectively, with $\mu_n = t^n$; we give their behaviour with respect to sequences belonging to one or more of the classes A, B_l, C_l or D defined as follows:

$\sigma \in A$ if and only if $a_n \equiv \sigma_n - \sigma_{n-1} = O(1)$;

$\sigma \in B_l$ if and only if $\{a_n\}$ is Borel-summable to l;

† In the proofs of theorems involving (H^*, μ_n) or (S^*, μ_n) methods, we shall assume that the associated function $\chi(t)$ is continuous at $t = 0$. This is no loss of generality, since τ^*_n, λ^*_n do not involve a. The statements of some of these theorems involve expressions of the type $\chi_a(1) - \chi_a(1-0)$, which we have however denoted by $\chi(1) - \chi(1-0)$ itself, since these expressions are independent of a. The '$\chi(t)$' of the proofs is not necessarily the same as the '$\chi(t)$' of the theorems.

$\sigma \in C_l$ if and only if $a_n \to l$ $(n \to \infty)$;

$\sigma \in D$ if and only if $\sigma_n = o(n^{\frac{1}{2}})$.

LEMMA 1. *The method $E(t)$, $0 < t < 1$, is absolutely regular for all sequences $\sigma \in D$.*

PROOF. Lorentz [11] has proved that if $E(t) = (e_{nk}(t))$, then

$$\lim_{n\to\infty} \sum_{k=0}^{n} \Omega(k) \, | e_{nk}(t) - e_{n,k+1}(t) | = 0 \tag{6}$$

for every function $\Omega(k) = o(k^{1/2})$ and each t in $(0, 1)$. Cooke [2] has shown that (6) implies the absolute regularity of $E(t)$ for all sequences $\{\sigma_k\}$, $\sigma_k = o(k^{1/2})$. Combining these two results, we get the lemma.

LEMMA 2. *The methods $E(t)$, $T(t)$ and $L(t)$ are, for $0 < t < 1$, equivalent to the Borel method for bounded sequences.*

LEMMA 3. *The methods $T(t)$, $L(t)$ are, for $0 < t < 1$ absolutely regular for $\sigma \in A \cap D$.*

Lemma 2 is due to Meyer-König ([13], Satz 25). Lemma 3 is a consequence of Lemmas 1 and 2; for, $T(t)$ and $L(t)$ are both equivalent to $E(t)$ for bounded sequences, and $E(t)$ sums $\sigma - \bar{\sigma}$ to 0 for all $\sigma \in D$.

LEMMA 4. *If $\sigma \in D$ then $\sigma \in B_0$; i.e. $\sigma_n = o(n^{1/2})$ implies that $\{a_n\} \equiv \{\sigma_n - \sigma_{n-1}\}$ is Borel-summable to* 0.

PROOF. We have

$$\frac{a_0 + a_1 + \dots + a_n}{n+1} = \frac{\sigma_n}{n+1} = o(n^{-1/2}).$$

By a well-known result due to Knopp ([6]; also Hardy [3], p. 213) it follows that $\{a_n\}$ is $E(t)$-summable to zero for every t in $(0, 1)$ and hence certainly $\sigma \in B_0$.

LEMMA 5. *Let $\{E_n(t, \sigma)\}$, $\{T_n(t, \sigma)\}$, $\{L_n(t, \sigma)\}$ denote, respectively the $E(t)$-, $T(t)$- and $L(t)$-transforms of $\sigma = \{\sigma_n\}$. Then for $0 < t < 1$,*

$$\lim_{n\to\infty} [E_n(t, \sigma) - E_{n-1}(t, \sigma)] = 0, \quad (\sigma \in D) \tag{7.1}$$

$$\lim_{n\to\infty} [T_n(t, \sigma) - T_{n-1}(t, \sigma)] = 0 \quad (\sigma \in A \cap D) \tag{7.2}$$

$$\lim_{n\to\infty} [L_n(t, \sigma) - L_{n-1}(t, \sigma)] = 0 \quad (\sigma \in A \cap D). \tag{7.3}$$

PROOF. We have, from the definition of the $E(t)$-transform, for all $n \geqslant 1$,

$$\begin{aligned} E_n(t, \sigma) &= \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \sigma_k \\ &= \sum_{k=0}^{n-1} \left\{ \binom{n-1}{k-1} + \binom{n-1}{k} \right\} t^k (1-t)^{n-k} \sigma_k \\ &= (1-t)\, E_{n-1}(t, \sigma) + E_{n-1}(t, \vec{\sigma}), \end{aligned}$$

i.e.

$$E_n(t, \sigma) - E_{n-1}(t, \sigma) = t\, E_{n-1}(t, \vec{\sigma} - \sigma). \tag{8}$$

A similar calculation gives also :

$$T_n(t, \sigma) - T_{n-1}(t, \sigma) = \frac{1}{t}\, T_n(t, \sigma - \overleftarrow{\sigma}), \tag{9}$$

$$L_n(t, \sigma) - L_{n-1}(t, \sigma) = \frac{1-t}{t}\, L_n(t, \vec{\sigma} - \sigma). \tag{10}$$

The relations (7.1), (7.2) and (7.3) follow now from (8), (9), (10) and Lemma 3.‡

5. The main theorems. We start with a theorem of Tauberian type for Hausdorff methods; the analogue for quasi-Hausdorff methods is given as Theorem 3 but restricted to bounded sequences. Theorems 1 and 3 may be compared with another result proved by me elsewhere ([14], Theorem 2).

THEOREM 1. *Let (H, μ_n) be a conservative Hausdorff method with $\lim \mu_n \neq 0$; let further $\{\sigma_n\} \equiv \sigma \in A \cap B_l$ be a sequence summable by the method (H, μ_n). Then $\sigma \in C_l$. (Of course $l = 0$ if $\sigma \in D$.)*

‡ Lemma 3 as well as the relations (7.2) and (7.3) are in fact true for all $\sigma \in D$ as I prove elsewhere [15].

PROOF. Let $\tau = \{\tau_n\}$ be the transform of σ by the (H, μ_n) method. Since (H, μ_n) is conservative, the relation (5) holds, so that we have for each fixed n,

$$\tau_n = \sum_{k=0}^{n} \int_0^1 \binom{n}{k} t^k (1-t)^{n-k} \sigma_k \, d\chi(t)$$

$$= \int_0^1 \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \sigma_k \, d\chi(t) \tag{11}$$

$$= \int_0^1 E_n(t, \sigma) \, d\chi(t), \tag{12}$$

where $\{E_n(t, \sigma)\}$ is the $E(t)$-transform of σ; the inversion of operations leading to (11) is obviously valid since the summation is only over a finite range $k = 0$ to $k = n$ and n is a fixed integer. From (12) we get for $n \geqslant 1$,

$$\tau_n - \tau_{n-1} = \int_0^1 [E_n(t, \sigma) - E_{n-1}(t, \sigma)] \, d\chi(t) \tag{13}$$

$$= \int_0^{\delta} \cdots + \int_{\delta}^{1-\delta} \cdots + \int_{1-\delta}^{1} \cdots \quad (0 < \delta < 1). \tag{14}$$

Now for each fixed n, $E_n(t, \sigma)$ is a polynomial in t, and we have

$$\lim_{t \to +0} E_n(t, \sigma) = \sigma_0 \quad \text{and} \quad \lim_{t \to 1-0} E_n(t, \sigma) = \sigma_n. \tag{15}$$

Keeping n fixed and letting $\delta \to 0$ in (14), we get, in virtue of (15) that, for $n \geqslant 1$,

$$\tau_n - \tau_{n-1} = \int_{+0}^{1-0} [E_n(t, \sigma) - E_{n-1}(t, \sigma)] \, d\chi(t) + $$
$$+ (\sigma_n - \sigma_{n-1}) [\chi(1) - \chi(1-0)]$$

$$\left.\begin{aligned} &= \int_{+0}^{1-0} t \, E_{n-1}(t, \vec{\sigma} - \sigma) \, d\chi(t) + \\ &\quad + (\sigma_n - \sigma_{n-1}) [\chi(1) - \chi(1-0)]. \end{aligned}\right\} \tag{16}$$

Since by hypothesis $\sigma \in A$, there is a number K such that

$$|t\, E_n(t, \bar{\sigma} - \sigma)| \leqslant |E_n(t, \bar{\sigma} - \sigma)| \leqslant K \tag{17.1}$$

for all n and $0 \leqslant t \leqslant 1$.

Further,

$$\lim_{n\to\infty} E_n(t, \sigma - \overrightarrow{\sigma}) = \lim_{n\to\infty} E_n(t, \sigma - \overleftarrow{\sigma}) = l, \text{ for } 0 < t < 1 \tag{17.2}$$

(by Lemmas 1, 2; and by Lemma 4, $l = 0$ if $\sigma \in D$). Hence the integrand in (16) tends to the continuous function lt for each t in $(0, 1)$ as $n \to \infty$. This yields, in view of (17.1) and the bounded variation of $\chi(t)$ in $(0, 1)$ that

$$\lim_{n\to\infty} \int_{+0}^{1-0} t\, E_n(t, \overrightarrow{\sigma} - \sigma)\, d\chi(t) = l \int_{+0}^{1-0} t\, d\chi(t)$$

$$= l\,[\mu_1 - \{\chi(1) - \chi(1-0)\}] = l_1, \text{ say.}$$

Now, if $\{\tau_n\}$ is convergent, then obviously $\tau_n - \tau_{n-1} = o(1)$ and therefore we get

$$l_1 + (\sigma_n - \sigma_{n-1})[\chi(1) - \chi(1-0)] = o(1).$$

Since $\chi(1) - \chi(1-0) \neq 0$ by hypothesis, this shows that

$$\sigma_n - \sigma_{n-1} \text{ tends to a limit } l_2 \text{ as } n \to \infty. \tag{18}$$

Since $\sigma \in B_l$ we have necessarily $l_2 = l$ by the regularity of the Borel method. This proves the theorem.

Remark. Let $H \equiv (H, \mu_n)$ be conservative and $\lim \mu_n \neq 0$. Then Theorem 1 shows that

H will sum no $\sigma = \{\sigma_n\}$ for which $\{\sigma_n - \sigma_{n-1}\}$ is bounded and not Borel-summable.

It is interesting to compare this result with another which has been proved by me elsewhere [14]:

H will sum no $\sigma = \{\sigma_n\}$ which is bounded, divergent and Borel-summable.

THEOREM 2.1 (a). *The conservative method* (H, μ_n) *has the property* (1), (2) *or* (5) *for the sequence* $\sigma \in A \cap B_l$ *if and only if*

$$\left.\begin{array}{l} \text{either } \sigma \in C_l \\ \text{or } \lim \mu_n \equiv \chi(1) - \chi(1-0) = 0. \end{array}\right\} \qquad (2.1)$$

(Of course, $l = 0$ *if* $\sigma \in D$*, by Lemma* 4.)

(b) *The conservative method* (H, μ_n) *has the properties* (8), (10) *and* (12) *for all sequences* $\sigma \in A \cap B_l$.

PROOF OF (a). Let $\tau = \{\tau_n\}$, $\bar{\tau} = \{\bar{\tau}_n\}$ be the transforms of the sequences σ, $\bar{\sigma}$ respectively, by the method (H, μ_n), and let $\sigma \in A \cap B_l$. Then as in the proof of Theorem 1, we can write

$$\tau_n - \bar{\tau}_n = \int_0^1 E_n(t, \sigma - \bar{\sigma})\, d\chi(t)$$

$$= (\sigma_0 - \bar{\sigma}_0)\,[\chi(+0) - \chi(0)] + \int_{+0}^{1-0} E_n(t, \sigma - \bar{\sigma})\, d\chi(t) +$$

$$+ (\sigma_n - \bar{\sigma}_n)\,[\chi(1) - \chi(1-0)].$$

By the relations (17.1), (17.2) and an argument as in the proof of Theorem 1 we get

$$\lim_{n\to\infty} \int_{+0}^{1-0} E_n(t, \sigma - \overleftarrow{\sigma})\, d\chi(t) = l\,[\chi(1-0) - \chi(+0)],$$

$$\lim_{n\to\infty} \int_{+0}^{1-0} E_n(t, \sigma - \overrightarrow{\sigma})\, d\chi(t) = -l\,[\chi(1-0) - \chi(+0)].$$

Therefore, $\{\tau_n - \bar{\tau}_n\}$ is convergent if and only if

$$(\sigma_n - \bar{\sigma}_n)\;[\chi(1) - \chi(1-0)]$$

converges. Let $\{\tau_n - \bar{\tau}_n\}$ converge, $\chi(1) - \chi(1-0) \neq 0$ and $\sigma_n - \sigma_{n-1} \to l_2$. Then $l_2 = l$ by the regularity of the Borel method. This proves part (a) of the theorem.

PROOF OF (b). The proof of (a) shows that (b) holds if and only if the condition (2.1) holds ; but each of (8), (10) and (12) implies

that $\{\tau_n\}$ is convergent and it now follows from Theorem 1 that (2.1) is satisfied (since $\sigma \in A \cap B_l$). This proves (b) and the theorem.

THEOREM 2.2 (a). *The conservative method* (H, μ_n) *has the property* (3), (6) *or* (7) *for the sequence* $\sigma \in A \cap B_l$ *if and only if the condition* (2.1) *and*

$$\chi(+0) - \chi(0) = l\,[\chi(1) - \chi(0)] = 0 \tag{2.2}$$

hold.

(b). *The conservative method* (H, μ_n) *has the property* (9), (13) *or* (14) *for the sequence* $\sigma \in A \cap B_l$ *if and only if the condition* (2.2) *holds.*

THEOREM 2.3 (a). *Let* $\sigma \in A \cap B_l$. *Then the conservative method* (H, μ_n) *has the property* (4) *for the sequence* σ *if the conditions* (2.1) *and* (2.2) *are satisfied, and only if the condition* (2.1) *and*

$$(\sigma_1 - \sigma_0)\,[\chi(+0) - \chi(0)] + l\,[\chi(1) - \chi(0)] = 0 \tag{2.3}$$

hold.

(b). *The conservative method* (H, μ_n) *has the property* (11) *for a sequence* $\sigma \in A \cap B_l$ *if and only if* (2.3) *holds.*

COROLLARY. *The conservative method* (H, μ_n) *has the property for all sequences* $\sigma \in A \cap B_l$ *if and only if the condition* (2.2) *holds; it has the property* (4) *for all sequences of the same class if and only if the conditions* (2.1) *and* (2.2) *are both satisfied.*

REMARKS. 1. For T-methods (and more generally, for multiplicative methods with multiplier $\rho \neq 0$), we have

$$\chi(+0) - \chi(0) = 0 \text{ and } \chi(1) - \chi(0) \neq 0.$$

Hence it follows from Theorem 2.2 and 2.3 that the T-method (H, μ_n) has the property (3), (4), (6), or (7) for the sequence $\sigma \in A \cap B_l$ if and only if the conditions (2.1) and $l = 0$ hold; and it has the property (9), (11), (13) or (14) for $\sigma \in A \cap B_l$ if and only if $l = 0$.

2. For (H, μ_n) methods which are multiplicative zero, the condition (2.2) is always satisfied. Hence such a method has the property (1), (2), ..., (6) or (7) for the sequence $\sigma \in A \cap B_l$ if and only if (2.1) holds; it has the properties (8)-(14) for all sequences $\sigma \in A \cap B_l$.

THEOREM 3. *Let (H^*, ν_n) be a conservative quasi-Hausdorff method, with* $\lim \nu_n \neq 0$; *let* $\sigma = \{\sigma_n\}$ *be a bounded sequence summable by the method* (H^*, ν_n). *Then* $\sigma \in C_0$.

PROOF. Let $\chi(t)$ be the function of bounded variation in (0, 1) generating the moment sequence $\{a, \nu_0, \nu_1, \dots\}$. As observed in § 3, it is clearly no loss of generality to assume that the function $\chi(t)$ is continuous at $t = 0$, and we assume that this condition is satisfied. Then if $\tau^* = \{\tau^*_n\}$ denote the (H^*, ν_n)-transform of $\sigma = \{\sigma_n\}$, we have

$$\left.\begin{aligned} \tau^*_n &= \sum_{k=n}^{\infty} \int_0^1 \binom{k}{n} t^{n+1}(1-t)^{k-n}\,\sigma_k\, d\chi(t) \\ &= \int_0^1 \sum \binom{k}{n} t^{n+1}(1-t)^{k-n}\,\sigma_k\, d\chi(t) \\ &= \int_0^1 T_n(t, \sigma)\, d\chi(t) \end{aligned}\right\} \tag{19}$$

the inversion of operations in (19) being justified by the boundedness of $\{\sigma_n\}$ and the conditions on $\chi(t)$. Therefore,

$$\tau^*_n - \tau^*_{n-1} = \int_0^1 [T_n(t, \sigma) - T_{n-1}(t, \sigma)]\, d\chi(t) \quad (n \geqslant 1). \tag{20}$$

Also, for each t in $[0, 1]$

$$\begin{aligned} |T_n(t, \sigma) - T_{n-1}(t, \sigma)| &\leqslant |T_n(t, \sigma)| + |T_{n-1}(t, \sigma)| \\ &\leqslant 2 \sup_n |\sigma_n| < M < \infty, \end{aligned} \tag{21}$$

where M is independent of n and t.

Further, by Lemma 5,

$$\lim_{n\to\infty} [T_n(t, \sigma) - T_{n-1}(t, \sigma)] = 0 \text{ for } 0 < t < 1. \tag{22}$$

By hypothesis $\{\tau^*_n\}$ is convergent, and hence, taking limits as $n \to \infty$ in both sides of the equation (20), we get

$$0 = \lim_{n\to\infty} (\tau^*_n - \tau^*_{n-1}) = \lim_{n\to\infty} \int_0^1 [T_n(t, \sigma) - T_{n-1}(t, \sigma)]\, d\chi(t)$$

$$= \lim_{n\to\infty} \left\{ \int_0^{+0} \cdots + \int_{+0}^{1-0} \cdots + \int_{1-0}^{1} \cdots \right\} d\chi(t)$$

$$= 0 + 0 + \lim_{n\to\infty} (\sigma_n - \sigma_{n-1}) \; [\chi(1) - \chi(1-0)].$$

The first term is 0, by (21) and the continuity of $\chi(t)$ at $t = 0$; the second term is 0 by (21), (22) and the bounded variation of $\chi(t)$, while the third follows from the same reasons and the fact that

$$\lim_{t\to 1-0} T_n(t, \sigma) = \sigma_n. \tag{23}$$

Since $\lim \nu_n \equiv \chi(1) - \chi(1-0) \neq 0$, it follows that

$$\sigma_n - \sigma_{n-1} \to 0$$

and the theorem is proved.

THEOREM 4.1. *The conservative method* (H^*, ν_n) *has the property* (1), (2), (5), (8), (10) *or* (12) *for the sequence* $\sigma \in A \cap B_l$ *if and only if the condition* (2.1) *holds.*

THEOREM 4.2. *The conservative method* (H^*, ν_n) *has the property* (3), (4), (6), (7), (9), (11), (13) *or* (14) *for a sequence* $\sigma \in A \cap B_l$ *if and only if the conditions* (2.1) *and* (2.2) *hold.*

PROOF OF THEOREM 4.1. In the notation of the proof of Theorem 3 we have, for $n \geqslant 1$,

$$\tau^*_n - \bar{\tau}^*_n = \sum \int \binom{k}{n} t^{n+1}(1-t)^{k-n}(\sigma_k - \bar{\sigma}_k)\, d\chi(t),$$

where we assume again, as we may, that $\chi(t)$ is continuous at $t = 0$. Since further $\sigma \in A$, we can invert the order of operations and get (for $n > 1$)

$$\tau^*_n - \bar{\tau}^*_n = \int \sum \binom{k}{n} t^{n+1}(1-t)^{k-n}(\sigma_k - \bar{\sigma}_k)\, d\chi(t)$$

$$= \left\{ \int_0^{+0} + \int_{+0}^{1-0} + \int_{1-0}^{0} \right\} \{T_n(t, \sigma - \bar{\sigma})\}\, d\chi(t)$$

$$= \int_{+0}^{1-0} T_n(t, \sigma - \bar{\sigma})\, d\chi(t) + (\sigma_n - \bar{\sigma}_n)\, [\chi(1) - \chi(1-0)].$$

by virtue of (23), the conditions on $\chi(t)$ and σ and the fact that $T_n(t, \sigma - \bar{\sigma})$ is a power-series in $(1-t)$ with a positive radius of convergence around $t = 1$ and hence is continuous at $t = 1$.

Also from Lemma 2 we have, for $0 < t < 1$,

$$\lim_{n\to\infty} T_n(t, \sigma - \bar{\sigma}) = +l \text{ or } -l \tag{24}$$

according as $\bar{\sigma}$ denotes $\overleftarrow{\sigma}$ or $\overrightarrow{\sigma}$. Since

$$| T_n(t, \sigma - \bar{\sigma}) | < M < \infty,$$

where M is independent of n and t in $[0,1]$, we get from (24)

$$\lim_{n\to\infty} \int_{+0}^{1-0} T_n(t, \sigma - \bar{\sigma})\, d\chi(t) = \pm l [\chi(1-0) - \chi(+0)]$$

exists. Hence $\{\tau_n - \bar{\tau}_n\}$ converges or diverges with

$$(\sigma_n - \bar{\sigma}_n)\, [\chi(1) - \chi(1-0)].$$

The conclusions of the theorem are now obvious.

Proof of Theorem 4.2. This follows from the proof of Theorem 4.1 in the same way as Theorems 2.2, 2.3 followed from the proof of Theorem 2.1.

Notes 1. In the proofs of Theorems 3, 4.1, 4.2, etc. involving (H^*, ν_n) methods, we make use of the remark in § 3 and assume that $\chi(+0) = \chi(0)$; otherwise, the expressions like (19) may not always exist.

2. For T-methods, the condition (2.2) reduces to $l = 0$; for methods which are multiplicative zero, the condition (2.2) is always satisfied.

As particular cases we have the following results.

Theorem 5. *The multiplicative method* (H, μ_n) *and the conservative method* (H^*, ν_n) *have all the properties* (1) *to* (14) *if they possess any one of them, for the class* $A \cap B_0$.

Proof. For Hausdorff methods, this is a consequence of Theorem 1 and the Remark 1 following the corollary to Theorem 2.3;

for quasi-Hausdorff methods, the result follows from Theorems 4.1, 4.2 and the Note 2 to Theorem 4.2.

REMARK. It is interesting to note that the results of the type given in Theorem 5 are not necessarily true if the class of sequences is not suitably restricted. For example, it has been shown by Kuttner ([7], p. 137) that, if there is no restriction on the sequences σ, it is possible for a regular Hausdorff method to be almost translative without being translative in the strict sense. It is not known whether Theorem 5 is true for a class of sequences larger than $A \cap B_0$.

THEOREM 6. *The multiplicative method* (H, μ_n) *is translative for all* $\sigma \in A \cap B_0$.

This is a consequence of Theorem 2.2.

COROLLARY. *Multiplicative Hausdorff methods have all the properties* (1) *to* (14) *for all* $\sigma \in A \cap B_0$.

THEOREM 7.1. *The conservative method* (H, μ_n) *is translative in the wide sense for bounded sequences. It is translative in the strict sense for all bounded sequences if and only if it is multiplicative.*

THEOREM 7.2. *The conservative method* (H^*, ν_n) *is translative in the strict sense for all bounded sequences.*

PROOF OF THEOREMS 7.1 AND 7.2. If σ is a bounded sequence, then certainly $\sigma \in D$, and Lemma 4 shows that $\sigma \in B_0$ and indeed $\sigma \in A \cap B_0$.

Theorem 7.1 now follows from Theorem 2.2, for we see from Theorem 1 that the condition (2.1) holds for every (bounded) $\sigma \in A \cap B_0$.

Theorem 7.2 follows similarly from Theorem 5.2, in virtue of Theorem 3.3 and the fact that $\sigma \in A \cap B_0$ and is bounded.

Theorems 7.1 and 7.2 for bounded sequences, which have been obtained here from our more general theorems, have also been

obtained by a different method (which is applicable to general types of matrices), by Kuttner [10] and Ramanujan [17].

THE METHODS (S^*, ω_n)

The transform $\{\lambda^*_n\}$ of the sequence $\{\sigma_n\}$ by the (conservative) method (S^*, ω_n) is defined by

$$\lambda^*_n = \sum_{k=0}^{\infty} \binom{n+k}{k} \Delta^k \omega_n \sigma_k \qquad (n \geqslant 0).$$

As remarked in § 3, we may assume without loss of generality, (writing $\omega_n = \mu_{n+1}$), that the function $\chi(t)$ generating the moment sequence $\{\mu_n\}$ is continuous at $t = 0$. We have then for every bounded sequence σ,

$$\lambda^*_n = \sum_{k=0}^{\infty} \int_0^1 \binom{n+k}{k} t^{n+1}(1-t)^k \sigma_k \, d\chi(t)$$

$$= \int_0^1 \sum \binom{n+k}{k} t^{n+1}(1-t)^k \sigma_k \, d\chi(t)$$

$$= \int_0^1 L_n(t, \sigma) \, d\chi(t).$$

We note that $L_n(t, \sigma)$ is a power-series in $(1-t)$ with radius of convergence $\rho \geqslant 1$, as may be easily verified. Hence

$$\lim_{t \to 1} L_n(t, \sigma) = \lim_{1-t \to 0} L_n(t, \sigma) = L_n(1, \sigma) = \sigma_0.$$

Also,

$$|L_n(t, \sigma)| < M < \infty, \tag{25}$$

where M is independent of n and t in [0,1].

We may therefore write, for all bounded σ,

$$\lambda^*_n = \int_0^1 L_n(t, \sigma) \, d\chi(t) = \int_0^{+0} \cdots + \int_{+0}^{1-0} \cdots + \int_{1-0}^{1} \cdots$$

$$= \int_{+0}^{1-0} L_n(t, \sigma) \, d\chi(t) + \sigma_0[\chi(1) - \chi(1-0)]. \tag{26}$$

We examine now the consequences of the relation (26).

THEOREM 8 (a). *The conservative method* (S^*, ν_n) *sums every Borel-summable bounded sequence.*

(b). *The method* (S^*, ν_n) *sums every Borel-summable bounded sequence to its Borel-limit if and only if it is a regular method.*

PROOF. Let $\sigma = \{\sigma_n\}$ be a bounded sequence Borel-summable to l. Using Lemma 2 and the relation (25) in the equation (26), we get

$$\lim_{n\to\infty} \lambda^*_n = l[\chi(1-0) - \chi(+0)] + \sigma_0[\chi(1) - \chi(1-0)]. \quad (27)$$

This proves (a).

The Borel method is regular, and hence considering convergent sequences only, we see that the necessity part of (b) is true; the sufficiency part of (b) has been proved by Ramanujan ([16], Theorem 13).

REMARK. Theorem 8 shows that the (S^*, ω_n) method behaves differently from the Hausdorff and quasi-Hausdorff methods; neither (a) nor (b) of Theorem 8 is true if we replace the (S^*, ω_n) method by a (H, μ_n) or (H^*, ν_n) method. In a sense, the (S^*, ω_n) method is more powerful than the other two; even in the special cases $L(t)$, $T(t)$, though these methods are, for $0 < t < 1$, equivalent to the Borel method for bounded sequences, we have the following result ([18], § 68, Satz VII):

If $0 < \alpha < 1$, *then* $T(\alpha) \subset L(\alpha)$; *and every Borel-summable sequence is* $L(\alpha)$-*summable (to the Borel-limit) whenever its* $L(\alpha)$ *transform exists.*

THEOREM 9.1. *The conservative method* (S^*, ν_n) *has the properties* (1), (2), (5), (8), (10) *and* (12) *for all* $\sigma \in A \cap B_l$.

THEOREM 9.2. *The conservative method* (S^*, ν_n) *has the property* (3), (6), (7), (9), (13) *or* (14) *for the sequence* $\sigma \in A \cap B_l$ *if and only if*

$$\left.\begin{aligned} \chi(1) - \chi(1-0) &= 0 \\ \textit{and} \qquad l[\chi(1-0) - \chi(+0)] &= 0. \end{aligned}\right\} \quad (9.2)$$

THEOREM 9.3. *The conservative method* (S^*, ν_n) *has the property* (4) *or* (11) *for a sequence* $\sigma \in A \cap B_l$ *if and only if*

$$(\sigma_1 - \sigma_0)[\chi(1) - \chi(1-0)] + l[\chi(1-0) - \chi(+0)] = 0 \quad (9.3)$$

and hence the property (4), *or* (11), *for all sequences* $\sigma \in A \cap B_l$ *if and only if* (9.2) *holds.*

PROOF OF THEOREMS 9.1-9.3. If $\{\lambda^*_n\}$, $\{\bar{\lambda}^*_n\}$ denote the (S^*, ν_n)-transforms of $\{\sigma_n\}$, $\{\bar{\sigma}_n\}$ respectively, then for $\sigma \in A$ we have (cf. proof of Theorem 4.1) for $n \geqslant 1$:

$$\lambda^*_n - \bar{\lambda}^*_n = \int_{+0}^{1-0} L_n(t, \sigma - \bar{\sigma})\, d\chi(t) + (\sigma_0 - \bar{\sigma}_0)[\chi(1) - \chi(1-0)].$$

Using Lemma 3 and a familiar argument used in the preceding theorems we get

$$\lim_{n \to \infty} \int_{+0}^{1-0} L_n(t, \sigma - \bar{\sigma})\, d\chi(t) = \pm l[\chi(1-0) - \chi(+0)]$$

exists (the positive or negative sign being taken according as σ denotes $\overleftarrow{\sigma}$ or $\overrightarrow{\sigma}$).

The conclusions of Theorems 9.1 to 9.3 now follow at once.

COROLLARY 1. *The regular method* (S, ν_n) *has the property* (3), (4), (6), (7), (9), (11), (13) *or* (14) *for the sequence* $\sigma \in A \cap B_l$ *if and only if* $l = 0$.

COROLLARY 2. *The conservative method* (S, ν_n) *has the property* (1), (2), ..., (13), or (14) *for all sequences* $\sigma \in A \cap D$ *(and in particular for all bounded sequences) if and only if*

$$\lim \nu_n \equiv \chi(1) - \chi(1-0) = 0.$$

PROOF. Corollary 1 is immediate from Theorems 9.2 and 9.3 since $\chi(1) = \chi(1-0) \neq \chi(+0)$ for all regular (S^*, ν_n) methods ([15], p. 211); Corollary 2 follows easily from Lemma 4 and Theorem 9.1-9.3.

6. Summability of sequences of 0's and 1's. The question of summability of sequences of 0's and 1's by Hausdorff methods has been treated by Lorentz [12], Hill [5] and Ramanujan [16] while the

same problem for quasi-Hausdorff methods has been treated by Ramanujan [16] whose results have been extended by me in [14]. Theorem 10 below gives a complete answer to the question and is a proper analogue to a similar result ([14], Theorem 3) on the summability of "almost convergent sequences", and more generally of Borel-summable bounded sequences.

THEOREM 10 (a). *The conservative method* (H, μ_n) *sums either*

(i) *almost all sequences of* 0*'s and* 1*'s*

or

(ii) *no divergent sequence of* 0*'s and* 1*'s*

according as, respectively, lim μ_n *is zero, or non-zero.*

(b). *The conclusion of* (a) *is valid also for a conservative quasi-Hausdorff method* (H^*, μ_{n+1}).

PROOF. When lim $\mu_n = 0$ the conclusions of both (a) and (b) are known ([5], [12], [14], [16]). If lim $\mu_n \neq 0$ and if $\{\sigma_n\}$ is any sequence of 0's and 1's summable by the Hausdorff or quasi-Hausdorff method in question, then it follows from Theorem 1 or Theorem 3 as the case may be, that lim $(\sigma_n - \sigma_{n-1}) = 0$ which is obviously impossible if σ is a divergent sequence of 0's and 1's. This proves the theorem.

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Ramanujan Institute of Mathematics, Madras
and
University of Madras

ON THE SUMMATION OF SINGULAR SERIES ASSOCIATED WITH CERTAIN QUADRATIC FORMS (I)

By K. ANANDA-RAU

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1. The quadratic forms which are mainly dealt with in this paper are

$$a(x^2+y^2)+c(z^2+t^2), \tag{1}$$

$$a(x^2+y^2)+2c(z^2+t^2), \tag{2}$$

where x, y, z, t are integer variables and the constant coefficients a, c are unequal odd positive primes. The summation of the singular series which arise when we consider the problem of finding the number of representations of a given positive integer n by one or other of these forms can be effected by a standard method introduced by Hardy and explained in detail by Kloosterman ([4], p. 148-152). The purpose of the present paper is to give an account of an alternative method of summation, which, though somewhat lengthy, may be of interest as it runs on substantially different lines. A further point of some interest is that in the course of the method adopted here we are led to properties of certain infinite series whose terms involve Ramanujan sums and which appear to have a natural place in this theory.

The singular series associated with the forms (1) and (2) involve, as is well known, certain Gauss sums ([4], p. 145). If h, k are integers, $h \geqslant 1$, $k \geqslant 1$, the Gauss sum $S(h, k)$ is defined by

$$S(h,k)=\sum_{\nu=1}^{k}\exp\frac{2\pi i h\nu^2}{k}.$$

If h, k are subject to the condition $(h, k)=1$, h not being necessarily less than k, then it can be shown by using known results ([4], p. 145) that†

† The value of $S^2(h, k)$ when k is a multiple of 4 and $(h, k) = 1$ is quoted by Kloosterman ([4], p. 145) as

$$2(-1)^{(m-1)/2}\,kihm, \tag{i}$$

where the positive integer m is the largest odd divisor of k. It is easy to verify that

$$S^2(h, k) = \begin{cases} ki^{k-1} \text{ if } k \text{ is odd}, \\ 0 \text{ if } k \equiv 2 \pmod 4, \\ 2\, ki^h \text{ if } k \equiv 0 \pmod 4). \end{cases} \tag{3}$$

If b is an integer greater than 1 and $(h, k) = 1$, it is not necessary that bh is prime to k. The calculation of the value of $S^2(bh, k)$ which is required for our purpose here can be made to depend on the results (3) by making use of a known property of Gauss sums, namely, that if $(h, k) = 1$, $(b, k) = u$, $k/u = k'$, $b/u = b'$ then ([4], p. 145),

$$S(bh, k) = u\, S(b'h, k'). \tag{4}$$

It will be noticed that the definitions of u, k', b' and the hypothesis $(h, k) = 1$ imply $(b'h, k') = 1$. Therefore the results (3) are applicable for the evaluation of $S^2(b'h, k')$, and so $S^2(bh, k)$ can be found by using (4).

2. In the singular series associated with the form (1) the terms involve $S^2(ah, k)\, S^2(ch, k)$ and in the singular series associated with the form (2) the terms involve $S^2(ah, k)\, S^2(2ch, k)$. We shall therefore evaluate these products of Gauss sums. The considerations which are initially developed here apply when a, c are any positive odd integers, not necessarily primes, and we shall assume for the present that a, c are subject to the less stringent condition of being merely any positive odd integers. We shall indicate in due course the stage at which we have to impose the further restriction that a, c should be unequal primes. In order to apply (3) and (4) we write

$$\left.\begin{aligned} &(a, k) = f,\ (c, k) = g,\ (2c, k) = w \\ &a/f = \alpha,\ c/g = \gamma,\ 2c/w = \eta \\ &k/f = F,\ k/g = G,\ k/w = W. \end{aligned}\right\} \tag{5}$$

when $m \equiv 1 \pmod 4$ the expression (i) equals $2kih$, and when $m \equiv 3 \pmod 4$ that expression equals $-2ki3h$, which in turn equals $2kih$, since $i2h = -1$ on account of h being prime to the even integer k and therefore odd. The value of $S^2(h, k)$ is therefore in any case equal to $2kih$.

We first compute $S^2(ah, k)$ and $S^2(ch, k)$. Recalling that a, c are odd we see that f, g are also odd, with the consequences that when k is odd, F, G are also odd, and that when k is even, $F \equiv G \equiv k \pmod 4$. Therefore if $(h, k) = 1$, on applying (3) and (4) we are led to

$$S^2(ah, k) = f^2\, S^2(\alpha h, F) = \begin{cases} f^2\, F i^{F-1} = f k i^{F-1} & \text{if } k \text{ is odd}, \\ 0 & \text{if } k \equiv 2 \pmod 4, \\ f^2 .\, 2F i^{\alpha h} = 2 f k i^{\alpha h} & \text{if } k \equiv 0 \pmod 4. \end{cases}$$

$$S^2(ch, k) = g^2\, S^2(\gamma h, G) = \begin{cases} g^2\, G i^{G-1} = g k i^{G-1} & \text{if } k \text{ is odd}, \\ 0 & \text{if } k \equiv 2 \pmod 4, \\ g^2 .\, 2\, G i^{\gamma h} = 2 g k i^{\gamma h} & \text{if } k \equiv 0 \pmod 4. \end{cases}$$

To evaluate $S^2(2ch, k)$ we observe that if k is odd or $k \equiv 2 \pmod 4$ we have W odd; if $k \equiv 4 \pmod 8$ we have, in view of c being odd $W \equiv 2 \pmod 4$; and if $k \equiv 0 \pmod 8$ we have $W \equiv 0 \pmod 4$. On account of these relations between k and W we obtain

$$S^2(2\,ch, k) = w^2\, S^2(\eta h, W) = \begin{cases} w^2\, W i^{W-1} = w k i^{W-1} & \text{if } k \text{ is odd}, \\ & \text{or if } k \equiv 2 \pmod 4, \\ 0 & \text{if } k \equiv 4 \pmod 8, \\ w^2 .\, 2\, W i^{\eta h} = 2\, w k i^{\eta h} & \text{if } k \equiv 0 \pmod 8. \end{cases}$$

Combining these results we get

$$S^2(ah, k)\, S^2(ch, k) = \begin{cases} fgk^2\, i^{F+G-2} & \text{if } k \text{ is odd}, \\ 0 & \text{if } k \equiv 2 \pmod 4, \\ 4\, fgk^2\, i^{(\alpha+\gamma)h} & \text{if } k \equiv 0 \pmod 4; \end{cases} \tag{6}$$

$$S^2(ah, k)\, S^2(2\,ch, k) = \begin{cases} fwk^2\, i^{F+W-2} & \text{if } k \text{ is odd}, \\ 0 & \text{if } k \equiv 2, 4, 6 \pmod 8, \\ 4\, fwk^2\, i^{(\alpha+\eta)h} & \text{if } k \equiv 0 \pmod 8. \end{cases} \tag{7}$$

The above results can be further simplified to advantage. We observe firstly that since a, c are odd, it follows that α, γ are odd,

and so $\alpha+\gamma$ is even; secondly, that when $k \equiv 0 \pmod 4$, h is odd (being prime to k), and so $h-1$ is even, with the consequence

$$(\alpha+\gamma)(h-1) \equiv 0 \pmod 4.$$

Therefore when $k \equiv 0 \pmod 4$ we have

$$i^{(\alpha+\gamma)h} = i^{\alpha+\gamma}. \tag{8}$$

Next when $k \equiv 0 \pmod 8$ we have $w \equiv 2 \pmod 4$ on account of c being odd; if follows that η is odd in that case; further α is odd and h is odd (being prime to k). Therefore when $k \equiv 0 \pmod 8$ we have

$$(\alpha+\eta)(h-1) \equiv 0 \pmod 4$$

$$i^{(\alpha+\eta)h} = i^{\alpha+\eta}. \tag{9}$$

Further we can show that when k is odd $F+G \equiv f+g \pmod 4$ and $F+W \equiv f+w \pmod 4$.

To do this we observe that if k is odd then f, g, w, F, G, W are all odd and therefore

$$(F-1)(f+1) \equiv 0 \pmod 4, \tag{10}$$

$$(G-1)(g+1) \equiv 0 \pmod 4, \tag{11}$$

$$(W-1)(w+1) \equiv 0 \pmod 4. \tag{12}$$

Adding the two congruences (10) and (11) and using the relation $Ff = Gg = k$ we get easily

$$(F+G)-(f+g)+2(k-1) \equiv 0 \pmod 4. \tag{13}$$

But since k is odd, $2(k-1) \equiv 0 \pmod 4$ and we get from (13)

$$F+G \equiv f+g \pmod 4.$$

Using (10) and (12) and the further relation $Ww = k$, we prove similarly that

$$F+W \equiv f+w \pmod 4.$$

It follows from these that

$$i^{F+G-2} = -i^{f+g}$$

$$i^{F+W-2} = -i^{f+w}.$$

Using these results and (8), (9) in (6), (7) we obtain

$$S^2(ah, k)\, S^2(ch, k) = \begin{cases} -fgk^2\, i^{f+g} \text{ if } k \text{ is odd,} \\ 0 \quad \text{if } k \equiv 2 \pmod 4, \\ 4\, fgk^2\, i^{\alpha+\gamma} \text{ if } k \equiv 0 \pmod 4; \end{cases} \tag{14}$$

$$S^2(ah, k)\, S^2(2ch, k) = \begin{cases} -fwk^2\, i^{f+w} \text{ if } k \text{ is odd,} \\ 0 \quad \text{if } k \equiv 2, 4, 6 \pmod 8, \\ 4\, fwk^2\, i^{\alpha+\eta} \text{ if } k \equiv 0 \pmod 8). \end{cases} \tag{15}$$

These evaluations (14), (15) show that $S^2(ah, k)\ S^2(ch, k)$ and $S^2(ah, k)\ S^2(2ch, k)$ are entirely independent of h. It is this favourable circumstance that contributes to the effective summation of singular series appearing in the sequal.

3. When a and c are fixed and k runs through the values 1, 2, ... we may regard f, g, w, α, γ, η as functions of k; and when we wish to emphasize this functional dependence we will write $f = f(k)$, $\alpha = \alpha(k)$, etc. We require to prove certain periodicity properties of these functions. As any common divisor of k and a is also a common divisor of $k + a$ and a and conversely, it is clear that $(a, k + a) = (a, k)$ and so $f(k)$ has the period a, from which property, it further follows that $\alpha(k)$ has the period a. For similar reasons $g(k)$ and $\gamma(k)$ have the period c. Therefore if we write

$$\left.\begin{aligned} \Phi_1(k) &= -fgi^{f+g} \\ \Phi_2(k) &= 4\, fgi^{\alpha+\gamma} \end{aligned}\right\} \tag{16}$$

it follows that $\Phi_1(k)$, $\Phi_2(k)$ are periodic with period ac.

Arguing similarly we find that $w(k)$, $\eta(k)$ have the period $2c$ and the functions $\Psi_1(k)$, $\Psi_2(k)$ defined by

$$\left.\begin{aligned}\Psi_1(k) &= -fwi^{f+w}\\ \Psi_2(k) &= 4fwi^{\alpha+\eta}\end{aligned}\right\} \tag{17}$$

are periodic with period $2ac$.

4. We are now in a position to deal with the summation of the singular series associated with the forms (1) and (2). We express these series in the first instance in terms of Ramanujan sums and the functions $\Phi_1(k)$, $\Phi_2(k)$, $\Psi_1(k)$, $\Psi_2(k)$, the Φ-functions appearing in the case of form (1) and the Ψ-functions in the case of form (2). We shall take the singular series relevant to these forms in order.

If n, k are integers, $n \geqslant 1$, $k \geqslant 1$ the Ramanujan sum $c_k(n)$ is defined by

$$c_k(n) = \sum_h \exp\left(-\frac{2\pi i h n}{k}\right),$$

where the index of summation h runs through the integers not greater than k and prime to k. It is known ([6], p. 181) that for a fixed n, $c_k(n)$ is bounded as $k \to \infty$, so that the series

$$\sum_{k=1}^{\infty} \frac{c_k(n)}{k^2} \tag{18}$$

is absolutely convergent. We have to introduce certain series related to the above. If v, b are integers such that $v \geqslant 1$, $1 \leqslant b \leqslant v$ we shall write

$$R_v(b) = \sum_{\substack{k \equiv b \\ (\text{mod } v)}} \frac{c_k(n)}{k^2},$$

the absolute convergence of this series following from that of (18).

Taking n to be the positive integer whose representation by the forms (1) and (2) is being considered, the singular series† related to the form (1) is ([4] p. 147)

† The definition of the phrase 'singular series' has not been uniformly the same in all writings. See the remarks on page 159 of [3] relating to the note on § 9·8 of Lecture IX. The definition adopted in this paper is convenient for our purpose here.

$$\frac{\pi^2}{ac}\, n\, \mathfrak{S}(n),$$

where

$$\mathfrak{S}(n) = \sum_{k=1}^{\infty} A_k(n),$$

$$A_k(n) = \frac{1}{k^4} \sum_{h} S^2(ah, k)\, S^2(ch, k) \exp\left(-\frac{2\pi i h n}{k}\right),$$

h taking the values satisfiying $(h, k) = 1$, $1 \leqslant h \leqslant k$.

In view of the results and definitions appearing above we have

$$A_k(n) = \begin{cases} (1/k^2)\, \Phi_1(k)\, c_k(n) \text{ if } k \text{ is odd}, \\ 0 \qquad\qquad \text{if } k \equiv 2 \pmod 4, \\ (1/k^2)\, \Phi_2(k)\, c_k(n) \text{ if } k \equiv 0 \pmod 4. \end{cases}$$

It therefore follows that

$$\mathfrak{S}(n) = \sum_{k \text{ odd}} \Phi_1(k) \frac{c_k(n)}{k^2} + \sum_{\substack{k \equiv 0 \\ (\text{mod } 4)}} \Phi_2(k) \frac{c_k(n)}{k^2}. \tag{19}$$

Let now ρ denote an odd number in the interval $1 \leqslant \rho < 2ac$ and ξ denote a multiple of 4 in the interval $1 \leqslant \xi \leqslant 4ac$. From the periodicity properties of $\Phi_1(k)$, $\Phi_2(k)$ established above, it follows that

$$\Phi_1(k) = \Phi_1(\rho) \text{ if } k \equiv \rho \pmod{2ac},$$

$$\Phi_2(k) = \Phi_2(\xi) \text{ if } k \equiv \xi \pmod{4ac}.$$

Using these results and writing for shortness $2ac = M$, $4ac = N$ we easily see that (19) leads to

$$\mathfrak{S}(n) = \sum_{\rho} \Phi_1(\rho) \sum_{\substack{k \equiv \rho \\ (\text{mod } M)}} \frac{c_k(n)}{k^2} + \sum_{\xi} \Phi_2(\xi) \sum_{\substack{k \equiv \xi \\ (\text{mod } N)}} \frac{c_k(n)}{k^2},$$

that is,

$$\mathfrak{S}(n) = \sum_{\rho} \Phi_1(\rho)\, R_M(\rho) + \sum_{\xi} \Phi_2(\xi)\, R_N(\xi), \tag{20}$$

where ρ, ξ run through the values satisfying

$$\left.\begin{array}{ll} \rho \equiv 1 (\mathrm{mod}\ 2), & 1 \leqslant \rho < 2ac, \\ \xi \equiv 0 (\mathrm{mod}\ 4), & 1 < \xi \leqslant 4ac. \end{array}\right\} \tag{21}$$

5. In order to proceed further with the evaluation of the series on the right of (20) we have to divide the sets of numbers ρ, ξ into several subsets, the manner of division depending on the divisors of a and c. This division into subsets is simplest when a is 1 or an odd prime and c is 1 or an odd prime. We shall set down the details here for the case when a, c are both odd primes different from each other. The modifications required to deal with the case when $(a-1)(c-1)=0$ will be found to be fairly easy. We shall suppose then that a, c are unequal odd primes in all that follows.

We divide the set of integers ρ satisfying the condition stated in (21) into four classes, the numbers in these classes being typified by X, Y, Z, V and having the following properties:

$$\left.\begin{array}{lll} (X, 2a) = 1, & (X, 2c) = 1, & 1 \leqslant X < 2ac \\ (Y, 2a) = 1, & (Y, 2c) = c, & 1 \leqslant Y < 2ac \\ (Z, 2a) = a, & (Z, 2c) = 1, & 1 \leqslant Z < 2ac \\ (V, 2a) = a, & (V, 2c) = c, & 1 \leqslant V < 2ac. \end{array}\right\} \tag{22}$$

It is plain that these four sets are mutually exclusive and exhaust the numbers ρ.

It easily follows from the above properties of X, Y, Z, V that

$$f(X) = 1, \quad g(X) = 1, \quad f(Y) = 1, \quad g(Y) = c,$$
$$f(Z) = a, \quad g(Z) = 1, \quad f(V) = a, \quad g(V) = c,$$

and therefore we have

$$\Phi_1(X) = 1, \qquad \Phi_1(Y) = -ci^{1+c}$$
$$\Phi_1(Z) = -ai^{1+a}, \quad \Phi_1(V) = -aci^{a+c}.$$

Using these in the first sum on the right of (20) we get

$$\sum_{\rho} \Phi_1(\rho)\, R_M(\rho) = \sum_{X} R_M(X) - ci^{1+c} \sum_{Y} R_M(Y) -$$

$$- ai^{1+a} \sum_{Z} R_M(Z) - aci^{a+c} \sum_{V} R_M(V). \tag{23}$$

We transform the second sum on the right of (20) taken over ξ in a similar manner. We divide the set of integers ξ satisfying the conditions stated in (21) into four classes, the numbers in these classes being typified by X', Y', Z', V' and having the following properties:

$$\left.\begin{aligned}
&(X', a) = 1,\ (X', c) = 1,\ X' \equiv 0 \pmod 4),\ 1 < X' \leqslant 4ac\\
&(Y', a) = 1,\ (Y', c) = c,\ Y' \equiv 0 \pmod 4),\ 1 < Y' \leqslant 4ac\\
&(Z', a) = a,\ (Z', c) = 1,\ Z' \equiv 0 \pmod 4),\ 1 < Z' \leqslant 4ac\\
&(V', a) = a,\ (V', c) = c.\ V' \equiv 0 \pmod 4),\ 1 < V' \leqslant 4ac.
\end{aligned}\right\} \tag{24}$$

Straightforward calculation now gives us

$$\Phi_2(X') = 4i^{a+c}, \qquad \Phi_2(Y') = 4ci^{1+a},$$

$$\Phi_2(Z') = 4ai^{1+c}, \qquad \Phi_2(V') = -4ac.$$

Using these to transform the second sum on the right of (20) as mentioned, and combining the result with (20) and (23) we get

$$\begin{aligned}
\mathfrak{S}(n) = {} & \Sigma\, R_M(X) - ci^{1+c}\, \Sigma\, R_M(Y) - \\
& - ai^{1+a}\, \Sigma\, R_M(Z) - aci^{a+c}\, R_M(V) + \\
& + 4i^{a+c}\, \Sigma\, R_N(X') + 4ci^{1+a}\, \Sigma\, R_N(Y') + \\
& + 4ai^{1+c}\, \Sigma\, R_N(Z') - 4ac\, \Sigma\, R_N(V').
\end{aligned} \tag{25}$$

6. In order to enable us to express $\mathfrak{S}(n)$ in terms of the divisors of n we require to express the sums $\Sigma\, R_M(X)$, etc. appearing above in terms of another type of series also involving Ramanujan sums. We shall define these series $T(v)$ and prove some lemmas required for our purpose. If v is an integer $\geqslant 1$, we write

$$T(v) = \sum_{(k,v)=1} \frac{c_k(n)}{k^2} \tag{26}$$

with the understanding (which is natural) that all integers $\geqslant 1$ are to be regarded as prime to 1, so that,

$$T(1) = \sum_{k=1}^{\infty} \frac{c_k(n)}{k^2}.$$

LEMMA 1. *Let H be an integer > 1 and $\mathfrak{h}$ a divisor of H. Then*

$$\sum_{\mathfrak{a}} R_H(\mathfrak{a}) = T(\mathfrak{h}),$$

where the index of summation $\mathfrak{a}$ takes all values satisfying $(\mathfrak{a}, \mathfrak{h}) = 1$, $1 \leqslant \mathfrak{a} \leqslant H$.

PROOF. If $\mathfrak{h} = 1$, the proof is immediate. For by the convention made above the numbers not greater than H and prime to 1 are $1, 2, \dots, H$, and so

$$\sum_{\mathfrak{a}} R_H(\mathfrak{a}) = \sum_{\mathfrak{a}=1}^{H} R_H(\mathfrak{a}) = \sum_{\mathfrak{a}=1}^{H} \sum_{\substack{k \equiv \mathfrak{a} \\ (\operatorname{mod} H)}} \frac{c_k(n)}{k^2} = \sum_{k=1}^{\infty} \frac{c_k(n)}{k^2} = T(1).$$

We will suppose then that $\mathfrak{h} > 1$. Let $\Lambda = H/\mathfrak{h}$. Let the numbers less than $\mathfrak{h}$ and prime to it be

$$j_1, j_2, \dots, j_q.$$

Now

$$T(\mathfrak{h}) = \sum_{(k, \mathfrak{h})=1} \frac{c_k(n)}{k^2}. \tag{27}$$

The index of summation k in the above series can be allowed to run through the values

$$k = m\mathfrak{h} + j_r,$$

where r runs through $1, 2, \dots, q$ and m runs through $0, 1, 2, \dots$ Further m itself may be allowed to take values given by

$$m = p\Lambda + l,$$

where l takes the values $0, 1, 2, \dots, (\Lambda - 1)$ and p takes the values $0, 1, 2, \dots$.

The values of k in the series on the right of (27) can therefore be taken as given by

$$k = (p\,\Lambda + l)\,\mathfrak{h} + j_r = pH + l\mathfrak{h} + j_r, \tag{28}$$

where p, l, r take the values mentioned. If now we keep r, l fixed for the moment and allow p to run through 0, 1, 2, ... and write

$$Q = pH + l\,\mathfrak{h} + j_r,$$

then

$$\sum_{p=0}^{\infty} \frac{c_Q(n)}{Q^2} = R_H(l\mathfrak{h} + j_r), \tag{29}$$

since Q runs through all the values $\equiv l\mathfrak{h} + j_r \pmod{H}$ and $l\mathfrak{h} + j \leqslant (\Lambda - 1)\,\mathfrak{h} + \mathfrak{h} = H$.

Summing the identity (29) over r, l and taking into account (27) and (28) we are therefore led to

$$T(\mathfrak{h}) = \sum_{l=0}^{\Lambda-1} \sum_{r=1}^{q} R_H(l\mathfrak{h} + j_r). \tag{30}$$

Now since $l\mathfrak{h} + j_r \leqslant H$, $\Lambda\mathfrak{h} + j_r > H$ and $l\mathfrak{h} + j_r \equiv j_r \pmod{\mathfrak{h}}$ we see that the numbers $l\mathfrak{h} + j_r$ constitute precisely the numbers $\mathfrak{a}$ prime to $\mathfrak{h}$ and not greater than H. It therefore follows from (30) that

$$T(\mathfrak{h}) = \sum_{\mathfrak{a}} R_H(\mathfrak{a})$$

as required to be proved.

Lemma 2. *Let a, c be different odd primes and $M = 2ac$. Let X, Y, Z, V run through the values prescribed in* (22). *Then*

$$\left.\begin{aligned} \Sigma\, R_M(X) &= T(2ac) \\ \Sigma\, R_M(Y) &= T(2a) - T(2ac) \\ \Sigma\, R_M(Z) &= T(2c) - T(2ac) \\ \Sigma\, R_M(V) &= T(2) - T(2a) - T(2c) + T(2ac). \end{aligned}\right\} \tag{31}$$

Proof. Let $\mathfrak{f}$ typify the numbers satisfying $(\mathfrak{f},\ 2a) = 1$, $1 \leqslant \mathfrak{f} < M$ and let $\mathfrak{g}$ typify the numbers satisfying $(\mathfrak{g},\ 2c) = 1$, $1 \leqslant \mathfrak{g} < M$. From (22) and (21) we see that the numbers X satisfy $(X,\ 2ac) = 1$, $1 \leqslant x < M$ and the numbers ρ satisfy $(\rho,\ 2) = 1$,

$1 \leqslant \rho < M$. We may therefore apply Lemma 1 with $H = M$ and the following combinations of $\mathfrak{h}$ and $\mathfrak{a}$:

$$\mathfrak{h} = 2ac, \quad \mathfrak{a} = X$$

$$\mathfrak{h} = 2a, \quad \mathfrak{a} = \mathfrak{f}$$

$$\mathfrak{h} = 2c, \quad \mathfrak{a} = \mathfrak{g}$$

$$\mathfrak{h} = 2, \quad \mathfrak{a} = \rho.$$

In the above, by $\mathfrak{a} = X$ is meant that for the set $\mathfrak{a}$ of Lemma 1 we take here the set X, with similar meanings for the symbols $\mathfrak{a} = \mathfrak{f}$, $\mathfrak{a} = \mathfrak{g}$, $\mathfrak{a} = \rho$.

The application of Lemma 1 to the four cases gives

$$\left.\begin{aligned} \Sigma\, R_M(X) &= T(2ac), \quad \Sigma\, R_M(\mathfrak{f}) = T(2a) \\ \Sigma\, R_M(\mathfrak{g}) &= T(2c), \quad \Sigma\, R_M(\rho) = T(2). \end{aligned}\right\} \tag{32}$$

Now, plainly the set $\mathfrak{f}$ is made up of the sets X, Y, the set $\mathfrak{g}$ is made up of the sets X, Z, and the set ρ is made up of the sets X, Y, Z, V. Therefore we have

$$\left.\begin{aligned} \Sigma\, R_M(\mathfrak{f}) &= \Sigma\, R_M(X) + \Sigma\, R_M(Y) \\ \Sigma\, R_M(\mathfrak{g}) &= \Sigma\, R_M(X) + \Sigma\, R_M(Z) \\ \Sigma\, R_M(\rho) &= \Sigma\, R_M(X) + \Sigma\, R_M(Y) + \Sigma\, R_M(Z) + \Sigma\, R_M(V). \end{aligned}\right\} \tag{33}$$

The solution of the seven simultaneous linear equations (32), (33) leads us easily to the assertions (31) of the lemma.

LEMMA 3. *Let* $\mathfrak{N} \equiv 0 \pmod 4$ *and* $\mathfrak{d}$ *an odd divisor of* $\mathfrak{N}$. *Then*

$$\sum_{\mathfrak{b}} R_{\mathfrak{N}}(\mathfrak{b}) = T(\mathfrak{d}) - T(2\mathfrak{d}) - \frac{c_2(n)}{4}\, T(2\mathfrak{d}),$$

where $c_2(n)$ *is the Ramanujan sum and the index of summation* $\mathfrak{b}$ *takes all values satisfying the conditions*

$$\mathfrak{b} \equiv 0 \pmod 4, \quad (\mathfrak{b}, \mathfrak{d}) = 1, \quad 1 < \mathfrak{b} \leqslant \mathfrak{N}.$$

PROOF. Since $\mathfrak{d}$ is a divisor of $\mathfrak{N}$ we have by Lemma 1,

$$T(\mathfrak{d}) = \sum_{\mathfrak{a}} R_{\mathfrak{N}}(\mathfrak{a}), \tag{34}$$

where $\mathfrak{a}$ takes all values satisfying the conditions $(\mathfrak{a}, \mathfrak{d}) = 1$, $1 \leqslant \mathfrak{a} \leqslant \mathfrak{N}$.

Now it is plain that the set $\mathfrak{a}$ may be regarded as being made up of the three classes of numbers, namely, those that are odd, those that are congruent to 2 (mod 4) and those that are multiples of 4. That is to say, the set $\mathfrak{a}$ is made up of the set $\mathfrak{b}$ appearing in the enunciation of the present lemma and the two sets whose members are typified by ω, 2θ with the following properties:

$$\left.\begin{array}{lll} (\omega, \mathfrak{d}) = 1, & (\omega, 2) = 1, & 1 \leqslant \omega < \mathfrak{N} \\ (2\theta, \mathfrak{d}) = 1, & (\theta, 2) = 1, & 1 < 2\theta \leqslant \mathfrak{N}. \end{array}\right\} \tag{35}$$

Therefore we have from (34)

$$T(\mathfrak{d}) = \Sigma\, R_{\mathfrak{N}}(\omega) + \Sigma\, R_{\mathfrak{N}}(2\theta) + \Sigma\, R_{\mathfrak{N}}(\mathfrak{b}). \tag{36}$$

Now the conditions $(\omega, \mathfrak{d}) = 1$, $(\omega, 2) = 1$ are equivalent to $(\omega, 2\mathfrak{d}) = 1$; and since $\mathfrak{d}$ is odd and $\mathfrak{N} \equiv 0 \pmod 4$ it is clear that $2\mathfrak{d}$ is a divisor of $\mathfrak{N}$. Further $1 \leqslant \omega \leqslant \mathfrak{N}$. Therefore by Lemma 1 we have

$$\Sigma\, R_{\mathfrak{N}}(\omega) = T(2\mathfrak{d}).$$

It follows from (36) that

$$\Sigma\, R_{\mathfrak{N}}(\mathfrak{b}) = T(\mathfrak{d}) - T(2\,\mathfrak{d}) - \Sigma\, R_{\mathfrak{N}}(2\theta).$$

For proving the present lemma we have therefore only to show that

$$\sum R_{\mathfrak{N}}(2\theta) = \frac{c_2(n)}{4}\, T(2\mathfrak{d}). \tag{37}$$

We require to apply a known property of Ramanujan sums, namely, that if $(\mathfrak{q}, \mathfrak{q}') = 1$ then ([3], p. 138)

$$c_{\mathfrak{q}\mathfrak{q}'}(n) = c_{\mathfrak{q}}(n)\, c_{\mathfrak{q}'}(n). \tag{38}$$

Now for any specific θ satisfying (35) we have

$$R_{\mathfrak{N}}(2\theta) = \sum \frac{c_k(n)}{k^2},$$

where in the sum on the right k takes all values $\equiv 2\theta \pmod{\mathfrak{N}}$, that is to say, the values given by

$$k = p\,\mathfrak{N} + 2\theta \quad (p=0, 1, 2, 3, \ldots). \tag{39}$$

Since $\mathfrak{N} \equiv 0 \pmod 4$, $\mathfrak{N}/2$ is even. Also θ is odd from (35). Therefore (39) can be written as

$$k = 2\left(p.\frac{\mathfrak{N}}{2} + \theta\right) = 2J, \tag{40}$$

where $J = p.\,\mathfrak{N}/2 + \theta$ is odd for $p = 0, 1, 2, \ldots$

Applying the property (38) of Ramanujan sums quoted we have therefore from (40)

$$\left.\begin{aligned} c_k(n) &= c_2(n)\, c_J(n) \\ \frac{c_k(n)}{k^2} &= \frac{c_2(n)}{4} \cdot \frac{c_J(n)}{J^2}. \end{aligned}\right\} \tag{41}$$

As k runs through all values $\equiv 2\,\theta \pmod{\mathfrak{N}}$ it is clear that J runs through all values $\equiv \theta \pmod{\mathfrak{N}/2}$. Therefore from (41) it follows that

$$\sum_{\substack{k \equiv 2\theta \\ (\text{mod}\,\mathfrak{N})}} \frac{c_k(n)}{k^2} = \frac{c_2(n)}{4} \cdot \sum_{\substack{J \equiv \theta \\ (\text{mod}\,\mathfrak{N}/2)}} \frac{c_J(n)}{J^2},$$

which is equivalent to

$$R_{\mathfrak{N}}(2\theta) = \frac{c_2(n)}{4}\, R_{\mathfrak{N}/2}(\theta).$$

Therefore

$$\sum R_{\mathfrak{N}}(2\theta) = \frac{c_2(n)}{4} \sum R_{\mathfrak{N}/2}(\theta), \tag{42}$$

the indices of summation on the two sides namely $2\,\theta$ and θ being governed by the condition (35).

Now to express

$$\Sigma\, R_{\mathfrak{N}/2}(\theta) \tag{43}$$

in terms of a series T we observe that since $\mathfrak{h}$ is an odd divisor of $\mathfrak{N}$ and $\mathfrak{N} \equiv 0 \pmod 4$ it follows that $2\,\mathfrak{h}$ is a divisor of $\mathfrak{N}/2$; further we have from (35) $(\theta, 2\,\mathfrak{h}) = 1$ and $1 \leqslant \theta \leqslant \mathfrak{N}/2$. Therefore we can apply Lemma 1 to the sum (43) and obtain

$$\Sigma\, R_{\mathfrak{N}/2}(\theta) = T(2\,\mathfrak{h}). \tag{44}$$

Combining (42) and (44) we are led to (37) and so the lemma is proved.

LEMMA 4. *Let a, c be different odd primes as usual, and let $N = 4ac$. Let X', Y', Z', V' run through the values prescribed in* (24). *For integers $p \geqslant 1$ let $\mathfrak{A}(p)$ be defined by*

$$\mathfrak{A}(p) = T(p) - T(2p) - \frac{c_2(n)}{4} T(2p).$$

Then we have

$$\left.\begin{aligned} &\Sigma R_N(X') = \mathfrak{A}(ac) \\ &\Sigma R_N(Y') = \mathfrak{A}(a) - \mathfrak{A}(ac) \\ &\Sigma R_N(Z') = \mathfrak{A}(c) - \mathfrak{A}(ac) \\ &\Sigma R_N(V') = \mathfrak{A}(1) - \mathfrak{A}(a) - \mathfrak{A}(c) + \mathfrak{A}(ac). \end{aligned}\right\} \tag{45}$$

PROOF. Let $\mathfrak{r}$ typify the numbers satisfying $(\mathfrak{r}, a) = 1$, $\mathfrak{r} \equiv 0 \pmod 4$, $1 < \mathfrak{r} \leqslant N$, and let $\mathfrak{s}$ typify the numbers satisfying $(\mathfrak{s}, c) = 1$, $\mathfrak{s} \equiv 0 \pmod 4$, $1 < \mathfrak{s} \leqslant N$. From (24) and (21) we see that the numbers X' satisfy $(X', ac) = 1$, $X' \equiv 0 \pmod 4$, $1 < X' \leqslant N$, and the numbers ξ satisfy $\xi \equiv 0 \pmod 4$, $1 < \xi \leqslant N$. Therefore we may apply Lemma 3 with $\mathfrak{N} = N$ and the following combinations of $\mathfrak{d}$ and $\mathfrak{b}$:

$$\mathfrak{d} = ac,\ \mathfrak{b} = X';\ \mathfrak{d} = a,\ \mathfrak{b} = \mathfrak{r};$$

$$\mathfrak{d} = c,\quad \mathfrak{b} = \mathfrak{s};\quad \mathfrak{d} = 1,\ \mathfrak{b} = \xi.$$

We then get

$$\left.\begin{aligned} &\Sigma R_N(X') = \mathfrak{A}(ac), \quad \Sigma R_N(\mathfrak{r}) = \mathfrak{A}(a), \\ &\Sigma R_N(\mathfrak{s}) = \mathfrak{A}(c), \qquad \Sigma R_N(\xi) = \mathfrak{A}(1). \end{aligned}\right\} \tag{46}$$

Now it is easily seen that the set $\mathfrak{r}$ is made up of the sets X', Y', the set $\mathfrak{s}$ is made up of the sets X', Z', and the set ξ is made up of the sets X', Y', Z', V'. Therefore we have

$$\left.\begin{aligned} &\Sigma R_N(\mathfrak{r}) = \Sigma R_N(X') + \Sigma R_N(Y') \\ &\Sigma R_N(\mathfrak{s}) = \Sigma R_N(X') + \Sigma R_N(Z') \\ &\Sigma R_N(\xi) = \Sigma R_N(X') + \Sigma R_N(Y') + \Sigma R_N(Z') + \Sigma R_N(V'). \end{aligned}\right\} \tag{47}$$

The solution of the seven simultaneous linear equations (46), (47) leads us to the assertions (45) of the present lemma.

7. If we now apply the results of Lemmas 2 and 4 so as to replace in (25) the sums $\Sigma R_M(X)$, etc. by the sums T, it is seen without difficulty that after some rearrangement and simplification we are led to an expression for $\mathfrak{S}(n)$ consisting of a sum of *constant* multiples of

$$T(1),\ T(2),\ T(a),\ T(2a),\ T(c),\ T(2c),\ T(ac),\ T(2ac),$$

the coefficients of these sums T depending only on a, c and on the parity of n, the latter dependence arising out of the occurrence of $c_2(n)$ in Lemma 4 and the equality as given by Ramanujan ([6], p. 180) namely $c_2(n) = \cos n\pi$. For example, if n is even and $a \equiv 1 \pmod 4$, $c \equiv 1 \pmod 4$ we find

$$\begin{aligned}\mathfrak{S}(n) = & -4ac\,T(1) + 6ac\,T(2) + 4c(a-1)\,T(a) - \\ & - 6c(a-1)\,T(2a) + 4a(c-1)\,T(c) - \\ & - 6a(c-1)\,T(2c) - 4(a-1)(c-1)\,T(ac) + \\ & + 6(a-1)(c-1)\,T(2ac). \end{aligned} \tag{48}$$

With a view to facilitate computation we employ two Tables, according to the parity of n, to obtain the expression of $\mathfrak{S}(n)$ in terms of the sums T. Writing

$$\mathfrak{S}(n) = \sum_{v} K(v)\,T(v),$$

where the index of summation v takes the eight values 1, 2, a, $2a$, c, $2c$, ac, $2ac$, the values of the coefficients $K(v)$ in the four cases, namely

$$a \equiv 1,\quad c \equiv 1,$$
$$a \equiv 3,\quad c \equiv 3,$$
$$a \equiv 1,\quad c \equiv 3,$$
$$a \equiv 3,\quad c \equiv 1,$$

(all the congruences being to modulus 4) are given in separate columns in the Tables, the heading of the column showing the case under consideration. For example, the expression (48) for $\mathfrak{S}(n)$ can be read off from the column headed $a \equiv 1$, $c \equiv 1$ in Table I.

TABLE I.

Form : $a(x^2+y^2)+c(z^2+t^2)$. Values of K when n is even.

	$a\equiv 1,\ c\equiv 1$	$a\equiv 3,\ c\equiv 3$	$a\equiv 1,\ c\equiv 3$	$a\equiv 3,\ c\equiv 1$
$K(1)$	$-4ac$	$-4ac$	$-4ac$	$-4ac$
$K(2)$	$6ac$	$6ac$	$4ac$	$4ac$
$K(a)$	$4c(a-1)$	$4c(a+1)$	$4c(a-1)$	$4c(a+1)$
$K(2a)$	$-6c(a-1)$	$-6c(a+1)$	$-4c(a-1)$	$-4c(a+1)$
$K(c)$	$4a(c-1)$	$4a(c+1)$	$4a(c+1)$	$4a(c-1)$
$K(2c)$	$-6a(c-1)$	$-6a(c+1)$	$-4a(c+1)$	$-4a(c-1)$
$K(ac)$	$-4(a-1)(c-1)$	$-4(a+1)(c+1)$	$-4(a-1)(c+1)$	$-4(a+1)(c-1)$
$K(2ac)$	$6(a-1)(c-1)$	$6(a+1)(c+1)$	$4(a-1)(c+1)$	$4(a+1)(c-1)$

TABLE II

Form : $a(x^2 + y^2) + c(z^2 + t^2)$. Values of K when n is odd.

	$a \equiv 1,\ c \equiv 1$	$a \equiv 3,\ c \equiv 3$	$a \equiv 1,\ c \equiv 3$	$a \equiv 3,\ c \equiv 1$
$K(1)$	$-4ac$	$-4ac$	$-4ac$	$-4ac$
$K(2)$	$4ac$	$4ac$	$2ac$	$2ac$
$K(a)$	$4c(a-1)$	$4c(a+1)$	$4c(a-1)$	$4c(a+1)$
$K(2a)$	$-4c(a-1)$	$-4c(a+1)$	$-2c(a-1)$	$-2c(a+1)$
$K(c)$	$4a(c-1)$	$4a(c+1)$	$4a(c+1)$	$4a(c-1)$
$K(2c)$	$-4a(c-1)$	$-4a(c+1)$	$-2a(c+1)$	$-2a(c-1)$
$K(ac)$	$-4(a-1)(c-1)$	$-4(a+1)(c+1)$	$-4(a-1)(c+1)$	$-4(a+1)(c-1)$
$K(2ac)$	$4(a-1)(c-1)$	$4(a+1)(c+1)$	$2(a-1)(c+1)$	$2(a+1)(c-1)$

8. In order to arrive at the final formulae giving $\mathfrak{S}(n)$ in terms of the divisors of n we require the two lemmas that follow, the first of them giving the relation between the sums T and the divisors of n.

LEMMA 5. *Let v be an integer $\geqslant 1$, χ_v the principal character* mod v *and $L(s, \chi_v)$ the Dirichlet L-function corresponding to χ_v. Further, n denoting the positive integer appearing in the definition* (26) *of $T(v)$, let $\Delta_v(n)$ be the sum of the positive divisors of n whose conjugates are prime to v. Then*†

$$n\,T(v)\,L(2, \chi_v) = \Delta_v(n). \tag{49}$$

PROOF. We have to make use of the known result ([3], p. 138)

$$\sum_{p|r} c_p(n) = \begin{cases} r \text{ if } r \text{ is a divisor of } n, \\ 0 \text{ if } r \text{ is not a divisor of } n. \end{cases} \tag{50}$$

Now since $\chi_v(p) = 1$ if $(p, v) = 1$, and $\chi_v(p) = 0$ otherwise, it is plain that

$$T(v) = \sum_{(k,v)=1} \frac{c_k(n)}{k^2} = \sum_{p=1}^{\infty} \frac{\chi_v(p)\, c_p(n)}{p^2},$$

and so, employing Dirichlet multiplication we have

$$T(v)\,L(2, \chi_v) = \sum_{p=1}^{\infty} \frac{\chi_v(p)\, c_p(n)}{p^2} \sum_{q=1}^{\infty} \frac{\chi_v(q)}{q^2} = \sum_{r=1}^{\infty} \frac{a_r}{r^2},$$

where

$$a_r = \sum_{pq=r} \chi_v(p)\, \chi_v(q)\, c_p(n) = \chi_v(r) \sum_{p|r} c_p(n).$$

It follows by using the known result (50) quoted above, that

$$T(r)\,L(2, \chi_v) = \sum_{r|n} \frac{\chi_v(r)}{r} = \sum_{\substack{r|n \\ (r,v)=1}} \frac{1}{r}. \tag{51}$$

† The proof of this lemma can be made to depend on more general results given by Carmichael ([1], p. 22). These latter results naturally require considerably more elaborate treatment. The simple direct proof given here for the lemma is appropriate for our present requirements.

If we write $n = rd$, so that r, d are conjugate to each other as divisors of n, it is plain that

$$\sum_{\substack{r|n\\(r,v)=1}} \frac{1}{r} = \frac{1}{n} \sum_{\substack{d|n\\(r,v)=1}} d = \frac{1}{n}\,\Delta_v(n). \tag{52}$$

From (51) and (52) the lemma follows†.

LEMMA 6. *If n is the product of two positive integers B, C, where $(B, C) = 1$, and E is the product of all the distinct prime factors of B, then*

$$\Delta_E(n) = \Delta_B(n) = B\,\sigma(C),$$

where, as usual, $\sigma(C)$ denotes the sum of the positive divisors of C.

PROOF. We first show that the necessary and sufficient condition in order that a divisor D of n should have its conjugate prime to B is that D should be of the form

$$D = B\,\delta, \tag{53}$$

where δ is a divisor of C.

The condition is sufficient: For if δ is a divisor of C and $C = \delta\delta'$, then since $(B, C) = 1$ we have $(B, \delta') = 1$. In other words, consider-

† It may be of interest to observe that if the real part of s is greater than 1 then Dirichlet multiplication applied as above gives

$$\sum_{p=1}^{\infty} \frac{1}{p^s} \sum_{k=1}^{\infty} \frac{c_k(n)}{k^s} = \sum_{r=1}^{\infty} \frac{a_r}{r^s},$$

where

$$a_r = \sum_{p|r} c_p(n),$$

and so using (50) we get

$$\zeta(s) \sum_{k=1}^{\infty} \frac{c_k(n)}{k^s} = \sum_{r|n} \frac{1}{r^{s-1}},$$

which is equivalent to Ramanujan's identity ([6], p. 185)

$$\frac{\sigma_{1-s}(n)}{\zeta(s)} = \sum_{k=1}^{\infty} \frac{c_k(n)}{k^s}.$$

There is also a simple proof of this identity given by Estermann. See ([2], pp. 194-5) or ([3], pp. 139-40).

ing $B\delta$, δ' as conjugate divisors of n, D defined by (53) has its conjugate δ' prime to B.

The condition is necessary : For suppose D is a divisor of n whose conjugate D' is prime to B. Then we have

$$n = DD' = BC.$$

Since $(D', B) = 1$, it follows that D' should be a divisor of C; and so $D = B\delta$, where $\delta = C/D'$ is a divisor of C.

From the preceding we conclude that

$$\Delta_B(n) = B \, \Sigma \, \delta,$$

where the sum of the right is taken over all the divisors δ of C. This is equivalent to $\Delta_B(n) = B\sigma(C)$.

Now if E is the product of all the distinct prime factors of B, it is clear that every integer which is prime to B is also prime to E, and conversely. Therefore $\Delta_E(n) = \Delta_B(n)$, and the proof of the lemma is complete.

9. We have seen that we can express $\mathfrak{S}(n)$ in terms of the sums T. We now apply in the first instance Lemma 5 to replace the T's by their expressions in terms of the sums Δ. To complete the calculations we require to know the values of $L(2, \chi_v)$ when v is 1, 2, a, $2a$, c, $2c$, ac, $2ac$. These values are available from standard results ([5], p. 423) and are as follows. Writing for shortness $\mathfrak{L}_v$ for $L(2, \chi_v)$ we have

$$\mathfrak{L}_1 = \pi^2/6, \quad \mathfrak{L}_2 = \pi^2/8$$

$$\left.\begin{aligned} \mathfrak{L}_a &= (a^2-1)\,\pi^2/6a^2, & \mathfrak{L}_{2a} &= (a^2-1)\,\pi^2/8a^2 \\ \mathfrak{L}_c &= (c^2-1)\,\pi^2/6c^2, & \mathfrak{L}_{2c} &= (c^2-1)\,\pi^2/8a^2 \\ \mathfrak{L}_{ac} &= \frac{\pi^2(a^2-1)(c^2-1)}{6\,a^2c^2}, & \mathfrak{L}_{2ac} &= \frac{\pi^2(a^2-1)(c^2-1)}{8\,a^2c^2}. \end{aligned}\right\} \quad (54)$$

Having obtained the expression of $\mathfrak{S}(n)$ in terms of the sums Δ as explained above, we apply Lemma 6 to replace the Δ's by their expressions in terms of the sums of certain groups of divisors of n. We write

$$n = 2^\beta a^\lambda c^\mu \mathfrak{m},$$

where β, λ, μ are integers $\geqslant 0$ and $(\mathfrak{m}, 2ac) = 1$. We consider the above expression of n as the product of two factors B, C chosen appropriately corresponding to the Δ we are dealing with, so that we are enabled to express by means of Lemma 6 the Δ in terms of a sum σ. For example to express $\Delta_{2a}(n)$ in terms of a σ we write $B = 2^\beta a^\lambda$, $C = c^\mu \mathfrak{m}$, so that $E = 2a$; and Lemma 6 gives us

$$\Delta_{2a}(n) = 2^\beta a^\lambda \sigma(c^\mu \mathfrak{m}). \tag{55}$$

When some or all of the indices β, λ, μ are zero a slight formal difficulty makes its appearance, which can however be overcome by observing that if h is a positive integer prime to n, and v is any positive integer, then $\Delta_{hv}(n) = \Delta_v(n)$. This is obvious, since $(n, h) = 1$ implies that every divisor of n is prime to h, and therefore every divisor of n whose conjugate is prime to v has that conjugate prime also to hv, and conversely (the converse being trivial). For example, if $\lambda = 0$ so that n is prime to a we have (with $v = 2$)

$$\Delta_{2a}(n) = \Delta_2(n) = 2^\beta \sigma(c^\mu m)$$

by taking $B = 2^\beta$, $C = c^\mu \mathfrak{m}$ in Lemma 6. That is,

$$\Delta_{2a}(n) = 2^\beta a^\lambda \sigma(c^\mu \mathfrak{m}),$$

since $\lambda = 0$, so that the formula (55) is still true.

10. We have completed the proof that the sum of the singular series $\pi^2 n \mathfrak{S}(n)/ac$ can be expressed in terms of the divisors of n. The final results of the computation appear in Table III below† classified according to the parity of n and according to the least positive residues of a, c to modulus 4. We write as mentioned earlier $n = 2^\beta a^\lambda c^\mu \mathfrak{m}$, where β, λ, μ are integers $\geqslant 0$ and $(\mathfrak{m}, 2ac) = 1$, and we introduce four functions $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, $\mathfrak{W}$ depending on the divisors of n and defined as follows :

† In making the computations for the entries in Table III (and also later in this paper) we have made use of the multiplicative property of the σ-function to separate out the factor $\sigma(2^\beta)$ wherever it is possible to do so, and replaced it by its value $2^{\beta+1} - 1$.

$$\mathfrak{X} = \sigma(a^\lambda c^\mu \mathfrak{m}) - \frac{a^{\lambda+1}}{a+1}\sigma(c^\mu \mathfrak{m}) - \frac{c^{\mu+1}}{c+1}\sigma(a^\lambda \mathfrak{m}) + \\ + \frac{a^{\lambda+1}c^{\mu+1}}{(a+1)(c+1)}\sigma(\mathfrak{m})$$

$$\mathfrak{Y} = \sigma(a^\lambda c^\mu \mathfrak{m}) - \frac{a^{\lambda+1}}{a-1}\sigma(c^n \mathfrak{m}) - \frac{c^{\mu+1}}{c-1}\sigma(a^\lambda \mathfrak{m}) + \\ + \frac{a^{\lambda+1}c^{\mu+1}}{(a-1)(c-1)}\sigma(\mathfrak{m}),$$

$$\mathfrak{Z} = -\sigma(a^\lambda c^\mu \mathfrak{m}) + \frac{a^{\lambda+1}}{a+1}\sigma(c^\mu \mathfrak{m}) + \frac{c^{\mu+1}}{c-1}\sigma(a^\lambda \mathfrak{m}) - \\ - \frac{a^{\lambda+1}c^{\mu+1}}{(a+1)(c-1)}\sigma(\mathfrak{m}),$$

$$\mathfrak{W} = -\sigma(a^\lambda c^\mu \mathfrak{m}) + \frac{a^{\lambda+1}}{a-1}\sigma(c^\mu \mathfrak{m}) + \frac{c^{\mu+1}}{c+1}\sigma(a^\lambda \mathfrak{m}) - \\ - \frac{a^{\lambda+1}c^{\mu+1}}{(a-1)(c+1)}\sigma(\mathfrak{m}).$$

The meanings of the entries in Table III are clear; for example, in the case $a \equiv 1$, $c \equiv 3$, the congruences being to modulus 4, the

TABLE III

	Sum of the singular series $\pi^2 n\mathfrak{S}(n)/ac$ relating to the form $a(x^2+y^2)+c(z^2+t^2)$.	
	n odd	n even
$a \equiv 1,\ c \equiv 1$	$8\mathfrak{X}$	$24\mathfrak{X}$
$a \equiv 3,\ c \equiv 3$	$8\mathfrak{Y}$	$24\mathfrak{Y}$
$a \equiv 1,\ c \equiv 3$	$8\mathfrak{Z}$	$8(2^{\beta+1}-3)\mathfrak{Z}$
$a \equiv 3,\ c \equiv 1.$	$8\mathfrak{W}$	$8(2^{\beta+1}-3)\mathfrak{W}$

sum of the singular series $\pi^2 n \mathfrak{S}(n)/ac$ is $8\mathfrak{Z}$ if n is odd and $8(2^{\beta+1}-3)\mathfrak{Z}$ if n is even.

11. We proceed to the evaluation of the singular series related to the form $a(x^2+y^2)+2c(z^2+t^2)$. As the procedure in this case is similar to that adopted in the case just treated, I shall give only the outlines of the steps without going into all the details. The singular series associated with the form $a(x^2+y^2)+2c(z^2+t^2)$ is $\dfrac{\pi^2}{2\,ac}\, n\, \mathfrak{S}'(n)$, where

$$\left.\begin{aligned} \mathfrak{S}'(n) &= \sum_{k=1}^{\infty} A_k'(n), \\ A'_k(n) &= \frac{1}{k^4}\sum_{h} S^2(ah, k)\, S^2(2\,ch, k)\exp\left(-\frac{2\pi i h n}{k}\right), \end{aligned}\right\} \tag{56}$$

h taking the values satisfying $(h, k)=1$, $1 \leqslant h \leqslant k$.

Using the results (15) and the definitions (17) of $\Psi_1(k)$, $\Psi_2(k)$ we have

$$A'_k(n) = \begin{cases} (1/k^2)\,\Psi_1(k)\, c_k(n) \text{ if } k \text{ is odd} \\ 0 \quad \text{if } k \equiv 2, 4, 6 \pmod 8 \\ (1/k^2)\,\Psi_2(k)\, c_k(n) \text{ if } k \equiv 0 \pmod 8). \end{cases} \tag{57}$$

Arguing as we did to obtain the expression (20) for $\mathfrak{S}(n)$ we see that if $2ac = M$, $8ac = P$ the periodicity properties of $\Psi_1(k)$, $\Psi_2(k)$ enable us to deduce from (56) and (57)

$$\mathfrak{S}'(n) = \sum_{\rho} \Psi_1(\rho)\, R_M(\rho) + \sum_{\tau} \Psi_2(\tau)\, R_P(\tau), \tag{58}$$

where ρ, τ run through the values satisfying

$$\left.\begin{aligned} \rho &\equiv 1 \pmod 2), \quad 1 \leqslant \rho < 2\,ac \\ \tau &\equiv 0 \pmod 8), \quad 1 < \tau \leqslant 8\,ac. \end{aligned}\right\} \tag{59}$$

We now divide the numbers ρ into four classes as before, the numbers in these classes being typified by X, Y, Z, V and satisfying

the conditions (22). We divide similarly the numbers $\mathfrak{j}$ into four classes, the numbers in these classes being typified by X'', Y'', Z'', V'' and having the following properties:

$$\left.\begin{array}{l}(X'', a) = 1,\ (X'', c) = 1,\ X'' \equiv 0 \pmod 8,\ 1 < X'' \leqslant 8\,ac \\ (Y'', a) = 1,\ (Y'', c) = c,\ Y'' \equiv 0 \pmod 8,\ 1 < Y'' \leqslant 8\,ac \\ (Z'', a) = a,\ (Z'', c) = 1,\ Z'' \equiv 0 \pmod 8,\ 1 < Z'' \leqslant 8\,ac \\ (V'', a) = a,\ (V'', c) = c,\ V'' \equiv 0 \pmod 8,\ 1 < V'' \leqslant 8\,ac.\end{array}\right\} \quad (60)$$

We obtain by straightforward calculation the values of $\Psi_1(X)$, $\Psi_1(Y)$, $\Psi_1(Z)$, $\Psi_1(V)$, $\Psi_2(X'')$, $\Psi_2(Y'')$, $\Psi_2(Z'')$, $\Psi_2(V'')$ in terms of a, c. Using these in (58) we obtain analogolusly to (25)

$$\begin{aligned}\mathfrak{S}'(n) = \Sigma\, R_M(X) - ci^{1+c}\, \Sigma\, R_M(Y) - ai^{1+a}\, \Sigma\, R_M(Z) - \\ - aci^{a+c}\, \Sigma\, R_M(V) + 8\,i^{a+c}\, \Sigma\, R_P(X'') + 8\,ci^{1+a}\, \Sigma\, R_P(Y'') + \\ + 8\,ai^{1+c}\, \Sigma\, R_P(Z'') - 8\,ac\, \Sigma\, R_P(V''). \quad (61)\end{aligned}$$

12. We have already found in Lemma 2 the expressions for $\Sigma\, R_M(X)$, $\Sigma\, R_M(Y)$, $\Sigma\, R_M(Z)$, $\Sigma\, R_M(V)$ in terms of sums T. To obtain the complete evaluation of $\mathfrak{S}'(n)$ in terms of sums T, it remains to express $\Sigma\, R_P(X'')$, $\Sigma\, R_P(Y'')$, $\Sigma\, R_P(Z'')$, $\Sigma\, R_P(V'')$ in terms of sums T. This is effected by the help of Lemmas 7, 8 that follow.

LEMMA 7. *Let* $\mathfrak{P} \equiv 0 \pmod 8$ *and* $\mathfrak{k}$ *an odd divisor of* $\mathfrak{P}$. *Then*

$$\sum_{\mathfrak{j}} R_{\mathfrak{P}}(\mathfrak{j}) = T(\mathfrak{k}) - \left\{1 + \frac{c_2(n)}{4} + \frac{c_4(n)}{16}\right\} T(2\,\mathfrak{k}),$$

where $c_2(n)$, $c_4(n)$ *are Ramanujan sums, and the index of summation* $\mathfrak{j}$ *takes all values satisfying*

$$\mathfrak{j} \equiv 0 \pmod 8,\ (\mathfrak{j}, \mathfrak{k}) = 1,\ 1 < \mathfrak{j} \leqslant \mathfrak{P}.$$

PROOF. Since $\mathfrak{k}$ is an odd divisor of $\mathfrak{P}$, and $\mathfrak{P}$ is a multiple of 4 we have by Lemma 3

$$T(\mathfrak{k}) - T(2\,\mathfrak{k}) - \frac{c_2(n)}{4}\, T(2\,\mathfrak{k}) = \sum_{\mathfrak{b}} R_{\mathfrak{P}}(\mathfrak{b}), \quad (62)$$

where the index of summation $\mathfrak{h}$ takes all values satisfying

$$\mathfrak{h} \equiv 0 \pmod 4),\ (\mathfrak{h}, \mathfrak{k}) = 1,\ 1 < \mathfrak{h} \leqslant \mathfrak{P}.$$

The set of numbers $\mathfrak{h}$ may plainly be regarded as being made up of the two classes of numbers namely, those that are congruent to 4 (mod 8) and those that are multiples of 8. That is to say, the set $\mathfrak{h}$ is made up of the set $\mathfrak{j}$ appearing in the enunciation of the present lemma and the set whose members are typified by 4ϕ with the following properties :

$$(4\phi, \mathfrak{k}) = 1,\ (\phi, 2) = 1,\ 1 < 4\phi \leqslant \mathfrak{P}. \tag{63}$$

Therefore we have

$$\Sigma\, R_{\mathfrak{P}}(\mathfrak{h}) = \Sigma\, R_{\mathfrak{P}}(\mathfrak{j}) + \Sigma\, R_{\mathfrak{P}}(4\phi),$$

and applying (62) we get

$$\sum R_{\mathfrak{P}}(\mathfrak{j}) = T(\mathfrak{k}) - T(2\,\mathfrak{k}) - \frac{c_2(n)}{4}\, T(2\,\mathfrak{k}) - \sum R_{\mathfrak{P}}(4\,\phi).$$

For proving the present lemma it only remains to show that

$$\sum R_{\mathfrak{P}}(4\,\phi) = \frac{c_4(n)}{16}\, T(2\,\mathfrak{k}). \tag{64}$$

Now for any specific ϕ satisfying (63) we have

$$R_{\mathfrak{P}}(4\,\phi) = \sum \frac{c_k(n)}{k^2},$$

where in the sum on the right the index of summation k takes all values $\equiv 4\phi \pmod{\mathfrak{P}}$, that is to say the values given by

$$k = p\,\mathfrak{P} + 4\phi \quad (p = 0, 1, 2, 3, \ldots). \tag{65}$$

Now $\mathfrak{P}/4$ is even since $\mathfrak{P} \equiv 0 \pmod 8$, and ϕ is odd from (63). Therefore (65) can be written as

$$k = 4\left(p.\frac{\mathfrak{P}}{4} + \phi\right) = 4\,\Gamma, \tag{66}$$

where $\Gamma = p.\ \mathfrak{P}/4 + \phi$ is odd for $p = 0, 1, 2, \ldots$. Therefore, since $(\Gamma, 4) = 1$ we have from (38)

$$c_k(n) = c_{4\Gamma}(n) = c_4(n)\, c_\Gamma(n)$$

$$\frac{c_k(n)}{k^2} = \frac{c_4(n)}{16} \cdot \frac{c_\Gamma(n)}{\Gamma^2}. \tag{67}$$

As k runs through all values $\equiv 4\phi \pmod{\mathfrak{P}}$ it is clear that Γ runs through all values congruent to $\phi \pmod{\mathfrak{P}/4}$. Therefore from (67) it follows that

$$\sum_{\substack{k\equiv 4\phi \\ (\mathrm{mod}\ \mathfrak{P})}} \frac{c_k(n)}{k^2} = \frac{c_4(n)}{16} \sum_{\substack{\Gamma\equiv\phi \\ (\mathrm{mod}\ \mathfrak{P}/4)}} \frac{c_\Gamma(n)}{\Gamma^2},$$

which is equivalent to

$$R_{\mathfrak{P}}(4\phi) = \frac{c_4(n)}{16} \cdot R_{\mathfrak{P}/4}(\phi).$$

Therefore

$$\sum R_{\mathfrak{P}}(4\phi) = \frac{c_4(n)}{16} \sum R_{\mathfrak{P}/4}(\phi) \tag{68}$$

the indices of summation on the two sides, namely 4ϕ, ϕ being governed by (63).

We proceed to express

$$\Sigma R_{\mathfrak{P}/4}(\phi) \tag{69}$$

in terms of a sum T. To do this we have only to observe that since $\mathfrak{k}$ is an odd divisor of $\mathfrak{P}$ and $\mathfrak{P} \equiv 0 \pmod 8$, it follows that $2\mathfrak{k}$ is a divisor of $\mathfrak{P}/4$; further we have from (63) $(\phi, 2\mathfrak{k}) = 1$ and $1 \leqslant \phi \leqslant \mathfrak{P}/4$. Therefore applying Lemma 1 to the sum (69) we get

$$\Sigma R_{\mathfrak{P}/4}(\phi) = T(2\,\mathfrak{k}). \tag{70}$$

Combining (68) and (70) we arrive at (64); and so the proof of the lemma is complete.

LEMMA 8. *Let $P = 8ac$, where a, c are, as usual, different odd primes. Let X'', Y'', Z'', V'' run through the values prescribed in* (60). *For integers $p \geqslant 1$ let $\mathfrak{B}(p)$ be defined by*

$$\mathfrak{B}(p) = T(p) - \left\{1 + \frac{c_2(n)}{4} + \frac{c_4(n)}{16}\right\} T(2p).$$

Then we have

$$\left.\begin{aligned} \Sigma R_P(X'') &= \mathfrak{B}(ac), \\ \Sigma R_P(Y'') &= \mathfrak{B}(a) - \mathfrak{B}(ac), \\ \Sigma R_P(Z'') &= \mathfrak{B}(c) - \mathfrak{B}(ac), \\ \Sigma R_P(V'') &= \mathfrak{B}(1) - \mathfrak{B}(a) - \mathfrak{B}(c) + \mathfrak{B}(ac). \end{aligned}\right\} \tag{71}$$

PROOF. Let $\mathfrak{v}$ typify the numbers satisfying

$$(\mathfrak{v}, a) = 1,\ \mathfrak{v} \equiv 0 \pmod 8,\ 1 < \mathfrak{v} \leqslant P,$$

and let $\mathfrak{w}$ typify the numbers satisfying

$$(\mathfrak{w}, c) = 1, \quad \mathfrak{w} \equiv 0 \pmod 8, \quad 1 < \mathfrak{w} \leqslant P.$$

From (60), (59) we see that the numbers X'' satisfy

$$(X'', ac) = 1, \quad X'' \equiv 0 \pmod 8, \quad 1 < X'' \leqslant P,$$

and the numbers τ satisfy

$$\tau \equiv 0 \pmod 8,\ 1 < \tau \leqslant P.$$

Therefore we may apply Lemma 7 with $\mathfrak{P} = P$ and the following combinations of $\mathfrak{k}$ and $\mathfrak{j}$:

$$\mathfrak{k} = ac,\ \mathfrak{j} = X''; \qquad \mathfrak{k} = a,\ \mathfrak{j} = \mathfrak{v}$$
$$\mathfrak{k} = c,\ \mathfrak{j} = \mathfrak{w}; \qquad \mathfrak{k} = 1,\ \mathfrak{j} = \tau.$$

We then get

$$\left.\begin{aligned} \Sigma R_P(X'') &= \mathfrak{B}(ac), & \Sigma R_P(\mathfrak{v}) &= \mathfrak{B}(a), \\ \Sigma R_P(\mathfrak{w}) &= \mathfrak{B}(c), & \Sigma R_P(\tau) &= \mathfrak{B}(1). \end{aligned}\right\} \tag{72}$$

Now it is easily seen that the set $\mathfrak{v}$ is made up of the sets X'', Y'', the set $\mathfrak{w}$ is made up of the sets X'', Y'' and the set τ is made up of the sets X'', Y'', Z'', V''. We have therefore

$$\left.\begin{aligned} \Sigma R_P(\mathfrak{v}) &= \Sigma R_P(X'') + \Sigma R_P(Y''), \\ \Sigma R_P(\mathfrak{w}) &= \Sigma R_P(X'') + \Sigma R_P(Z''), \\ \Sigma R_P(\tau) &= \Sigma R_P(X'') + \Sigma R_P(Y'') + \Sigma R_P(Z'') + \Sigma R_P(V''). \end{aligned}\right\} \tag{73}$$

The solution of the seven simultaneous linear equations (72), (73) leads us to the assertions (71) of the lemma.

13. Now reverting to (61) we see that the results of Lemmas 2, 8 enable us to express $\mathfrak{S}'(n)$ in terms of the sums

$$T(1),\ T(2),\ T(a),\ T(2a),\ T(c),\ T(2c),\ T(ac),\ T(2ac).$$

We have only to apply Lemmas 5, 6 as previously to obtain the expression of $\mathfrak{S}'(n)$ in terms of the divisors of n. The final results giving the sum of the singular series $\pi^2 n\, \mathfrak{S}'(n)/2ac$ are consolidated in Table IV below. The factorization $n = 2^\beta a^\lambda c^\mu \mathfrak{m}$ with $(\mathfrak{m}, 2ac) = 1$ is employed as before. The functions $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \mathfrak{W}$ are the ones defined earlier and used in Table III. Owing to the appearance of $c_4(n)$ in Lemmas 7 and 8 we get different results in the three cases, namely, when n is odd, when $n \equiv 2 \pmod 4$ and when $n \equiv 0 \pmod 4$. This is due to $c_4(n)$ being equal to $2 \cos n\pi/2$, and the value of $c_2(n)$ being $\cos n\pi$ according to the table given by Ramanujan ([6], p. 180). The explanation of the entries in Table IV is analogous to that given for the entries in Table III, and now relates to the sum $\pi^2 n\, \mathfrak{S}'(n)/2ac$.

The congruences for a, c appearing in the first column of Table IV are to modulus 4 as before.

TABLE IV

	Sum of the singular series $\pi^2 n\, \mathfrak{S}'(n)/2ac$ relating to the form $a(x^2+y^2)+2c(z^2+t^2)$.		
	n odd	$n \equiv 2 \pmod 4$	$n \equiv 0 \pmod 4$
$a \equiv 1,\ c \equiv 1$	$4\mathfrak{X}$	$8\mathfrak{X}$	$24\mathfrak{X}$
$a \equiv 3,\ c \equiv 3$	$4\mathfrak{Y}$	$8\mathfrak{Y}$	$24\mathfrak{Y}$
$a \equiv 1,\ c \equiv 3$	$4\mathfrak{Z}$	$8\mathfrak{Z}$	$8(2^\beta - 3)\,\mathfrak{Z}$
$a \equiv 3,\ c \equiv 1$	$4\mathfrak{W}$	$8\mathfrak{W}$	$8(2^\beta - 3)\,\mathfrak{W}$

14. The method used in the present paper to effect the summation of the singular series relating to the forms $a(x^2+y^2)+c(z^2+t^2)$, $a(x^2+y^2)+2c(z^2+t^2)$ when a, c are different odd primes is equally applicable for the evaluation of the sum of the singular series relating to the forms

$$(x^2+y^2)+c(z^2+t^2) \tag{74}$$

$$(x^2+y^2)+2c(z^2+t^2) \tag{75}$$

$$x^2+y^2+z^2+t^2 \tag{76}$$

$$(x^2+y^2)+2(z^2+t^2) \tag{77}$$

$$a(x^2+y^2)+2(z^2+t^2) \tag{78}$$

the c in (74), (75) and the a in (78) being odd primes. The modifications that have to be made in the preceding analysis to deal with these five cases are fairly easy. Care has to be taken, however, not to use the final formulae given above, and insert in the plausible ones of them $a=1$ to obtain the results for the forms (74), (75), $a=c=1$ to obtain the result for the forms (76), (77) and $c=1$ to obtain the result for the form (78). Such a procedure will lead to erroneous results, as some of the intermediate steps used in the preceding work do not hold when a or c or both take the value 1. For example, some or all of the evaluations (54) are not true when a or c or both have the value 1. To deal with the cases relating to the forms (74) to (78), the question of what modifications are required has to be gone into in some detail. The work however is quite straightforward, and we are led without difficulty to the results of Kloosterman ([4], pp. 151-52) in the case of forms (74), (75), to the result agreeing with the work of Jacobi in the case of the form (76), and to the result agreeing with the work of Liouville ([4], p. 144) in the case of the form (77). The singular series relating to the form (78) when a is any odd prime does not seem to have been worked out before. Writing $n = 2^{\beta} a^{\lambda} \mathfrak{m}$, where β, λ are integers $\geqslant 0$, and $(\mathfrak{m}, 2a)=1$ and defining $\mathfrak{X}'$, $\mathfrak{W}'$ by

$$\mathfrak{X}' = \sigma(a^{\lambda}\mathfrak{m}) - \frac{a^{\lambda+1}}{a+1}\,\sigma(\mathfrak{m}),$$

$$\mathfrak{W}' = -\,\sigma(a^{\lambda}\mathfrak{m}) + \frac{a^{\lambda+1}}{a-1}\,\sigma(\mathfrak{m}),$$

we have the following results giving the sum of the singular series $\pi^2 n\mathfrak{S}''(n)/2a$ related to the form (78), $\mathfrak{S}''(n)$ being what $\mathfrak{S}'(n)$ becomes when c is put equal to 1 in (56).

When $a \equiv 1 \pmod 4$ we have

$$\frac{\pi^2}{2a}\, n\, \mathfrak{S}''(n) = \begin{cases} 4\,\mathfrak{X}' \text{ if } n \text{ is odd} \\ 8\,\mathfrak{X}' \text{ if } n \equiv 2 \pmod 4 \\ 24\,\mathfrak{X}' \text{ if } n \equiv 0 \pmod 4. \end{cases}$$

When $a \equiv 3 \pmod 4$ we have

$$\frac{\pi^2}{2a}\, n\, \mathfrak{S}''(n) = \begin{cases} 4\,\mathfrak{W}' \text{ if } n \text{ is odd} \\ 8\,\mathfrak{W}' \text{ if } n \equiv 2 \pmod 4 \\ 8\,(2^{\beta} - 3)\,\mathfrak{W}' \text{ if } n \equiv 0 \pmod 4. \end{cases}$$

15. I may make a final observation. It will be seen that for the truth of the identities (20) and (58) the requirement is that a, c should be odd integers, not necessarily primes. A reference to this was made early in this paper. The necessity of imposing the condition that a, c should be primes arose essentially owing to the difficulty in dividing the sets ρ, ξ, τ into subsets which could conveniently be handled in the subsequent work. There are however several cases in which one or both of a, c are composite or one of a, c is 1 and the other composite, where it is possible to employ a more elaborate treatment of the sub-division of the sets ρ, ξ, τ so as to effect completely the summation of the relevant singular series. I hope to be able to consider these extensions on a future occasion.

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29, Boag Road
Madras 17

REMARKS ON A FUNCTIONAL EQUATION

By BERTHOLD SCHWEIZER

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IN a recent paper [1], V. Ganapathy Iyer studied the problem of finding integral functions, f, such that for any integral function g and any complex number a, the functional equation

$$f(\mathrm{az}) = f(\mathrm{z}) . g(\mathrm{z}) \tag{1}$$

holds for all complex numbers z. He completely determined the class of all integral functions, f, satisfying (1) when $|\mathrm{a}| \neq 1$ and found a number of solutions when $|\mathrm{a}| = 1$. But as is often the case in the study of functional equations, the functional form of the solution was obtained by a specialized argument—that is to say, it was not derived by means of any general technique.

In [2] and [3], M. A. McKiernan has studied a theory of iterator series, i.e. a theory dealing with the expansion of one function in a series of iterates of another, and has applied this theory to functional equations. For example, he has shown that, for any two functions g and h, a solution of the functional equation*

$$fh - f = g \tag{2}$$

is given by

$$f = -\sum_{n=0}^{\infty} g\, h^{[n]} \tag{3}$$

provided this series converges. Here $h^{[n]}$ is the n'th iterate of h, i.e. $h^{[0]} = j, h^{[1]} = h, \ldots, h^{[n]} = h^{[n-1]} h$.

The purpose of this note is to point out that the method developed by McKiernan is directly applicable to (1) and leads

* We follow the notational conventions of [4]. Lower case italics, such as $f, g, h, \ldots$, denote functions and lower case roman letters such as a, b, c, ..., denote numbers. The sum of f and g is denoted by $f+g$, the product by $f \cdot g$, and the result of substituting g into f by fg. Moreover, j is the identity function, that is to say $j(\mathrm{z})=\mathrm{z}$ for any complex number z; also $jf=fj=f$, for any function f.

quickly and easily to the solutions obtained by V. Ganapathy Iyer for the case $|a| \neq 1$. We proceed as follows : First of all, in the notation of [4], (1) becomes

$$f(aj) = f \cdot g. \tag{4}$$

Therefore,

$$\log f(aj) - \log f = \log g. \tag{5}$$

Hence replacing f by $\log f$, g by $\log g$, and h by aj in (3), and making use of the fact that the n'th iterate of aj is $a^n j$, we have formally,

$$\log f = - \sum_{n=0}^{\infty} \log g(a^n j) \tag{6}$$

whence we obtain

$$f = \prod_{n=0}^{\infty} \frac{1}{g(a^n j)}. \tag{7}$$

If $g(z) \neq 0$ for all z, then $1/g$ is an integral function ; and if $g(0) = 1$, we may write $1/g = 1 + j \cdot u$, where u is an integral function. Then

$$f(z) = \prod_{n=0}^{\infty} [1 + a^n z . u(a^n z)]. \tag{8}$$

This infinite product converges for any z whenever $0 < |a| < 1$, whence (7) defines a function which is readily seen to be a solution of (4). This is the result obtained by V. Ganapathy Iyer. To settle the case $g(0) \neq 1$, his arguments are now used.

In order to obtain a solution of (1) when $|a| > 1$, following [1], we transform (4) into

$$f = f\left(\frac{1}{a} j\right) \cdot g\left(\frac{1}{a} j\right) \tag{9}$$

which, using the same method as before, yields,

$$f = \prod_{n=1}^{\infty} g\left(\frac{1}{a^n} j\right). \tag{10}$$

Again, if $g(0) = 1$, we set $g = 1 + j \cdot v$ and obtain an infinite product similar to (8) converging for $|a| > 1$. The case $g(0) \neq 1$ is again argued as in [1].

The case $|a| = 1$ requires the special considerations given in [1]. For, from the above analysis we see that the situation is similar to that of a power series on its circle of convergence. Indeed, it can be shown that when $|a| = 1$ we are on the circle of convergence of the iterator series corresponding to (7) and (10), [3].

NOTE ADDED IN PROOF. The functional equational (2) and some of its generalizations have recently been studied in a series of interesting papers by M. Kuczma et al. [See, for example, M. Kuczma, On the functional equation $\phi(x) + \phi[f(x)] = F(x)$, *Annales Polonici Mathematici*, VI (1959) 281-287; J. Kordylewski and M. Kuczma, On the functional equation $F(x, \phi(x), \phi[f_1(x)], \dots, \phi[f_n(x)]) = 0$, *Ibid.*, VIII (1960), 55-60.]

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University of Arizona
Tucson

THEOREMS IMPLYING THE NON-VANISHING OF $\Sigma\chi(m)\,m^{-1}$ FOR REAL RESIDUE-CHARACTERS

By PAUL T. BATEMAN

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1. Introduction. Throughout this paper let us suppose that h is a given real-valued function on the positive integers and that related functions S and f on the positive integers are defined as follows:

$$S(n) = \sum_{m=1}^{n} h(m), \quad f(n) = \sum_{m/n} h(m). \tag{1.1}$$

(Thus S could be defined by the formal power series identity

$$\sum_{k=0}^{\infty} X^k \sum_{m=1}^{\infty} h\,(m)\,X^m = \sum_{n=1}^{\infty} S\,(n)\,X^n$$

and f by the formal Dirichlet series identity

$$\sum_{k=1}^{\infty} k^{-s} \sum_{m=1}^{\infty} h\,(m)\,m^{-s} = \sum_{n=1}^{\infty} f\,(n)\,n^{-s}).$$

The purpose of this paper is to prove the following theorems and other similar results.

THEOREM A. *If $S(n) \geqslant -K$ for all n, if $f(n) \geqslant 0$ for all n, and if $\sum_{n=1}^{\infty} f(n) > \frac{1}{2}K$, then*

$$\liminf_{n\to\infty} \sum_{m=1}^{n} h\,(m)\,m^{-1} > 0.$$

THEOREM B. *If $S(n) \geqslant -K$ for all n, where $K \geqslant 0$, if $f(n) \geqslant 0$ for all n, and if $f(n) \geqslant 1$ whenever n is a square, then*

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$$\liminf_{n\to\infty} \sum_{m=1}^{n} \frac{h(m)}{m} \geqslant \frac{32}{75K+150}.$$

An immediate corollary of either of these theorems is Dirichlet's famous lemma (crucial for the theory of primes in arithmetic progressions) that if χ is a real non-principal residue-character, then $\sum_{m=1}^{\infty} \chi(m)\, m^{-1} > 0$. In addition, Theorem B is an improvement of a result of Čudakov and Rodosskiĭ [1] on what they call generalized characters. Although our results are not deep, they may help to explain why it is possible to give elementary proofs of Dirichlet's lemma.

In case S is bounded above as well as below (in particular, if h is a real non-principal residue-character), the series $\Sigma\, h(m)\, m^{-1}$ converges. Further, since in this case

$$\lim_{x\to+\infty} x^{-1} \sum_{m\leqslant x} h(m) = 0, \quad \limsup_{x\to+\infty} x^{-1} \sum_{m\leqslant x} |h(m)| < +\infty,$$

it follows by a simple argument or by an elementary theorem of Axer [4, 20] that

$$\lim_{x\to+\infty} \frac{1}{x} \sum_{m\leqslant x} \left(\frac{x}{m} - \left[\frac{x}{m}\right]\right) h(m) = 0.$$

Inasmuch as

$$\frac{1}{x} \sum_{n\leqslant x} f(n) = \frac{1}{x} \sum_{m\leqslant x} \left[\frac{x}{m}\right] h(m)$$
$$= \sum_{m\leqslant x} \frac{h(m)}{m} - \frac{1}{x} \sum_{m\leqslant x} \left(\frac{x}{m} - \left[\frac{x}{m}\right]\right) h(m),$$

we therefore have in this case

$$\liminf_{x\to+\infty} \sum_{m\leqslant x} \frac{h(m)}{m} = \sum_{m=1}^{\infty} \frac{h(m)}{m} = \lim_{x\to+\infty} \frac{1}{x} \sum_{n\leqslant x} f(n).$$

Hence, if S is bounded above, the conclusion of our theorems may be modified by replacing the left-hand side by either of these alternative expressions.

In the particular case of real non-principal residue-characters there are two standard elementary arguments for proving Dirichlet's lemma, both originally due to Mertens. One of these [5, 7, 9, 13, 14, 18, 19] uses the sum $\sum_{n\leqslant x} f(n)\, n^{-\frac{1}{2}}$, while the other [3, 6, 8, 11, 15] uses the first-order Riesz mean

$$\sum_{n\leqslant x}\left(1-\frac{n}{x}\right)f(n).$$

Čudakov and Rodosskiĭ pointed out that for the purpose of obtaining more general results it is advisable to use instead the second-order Riesz mean

$$\sum_{n\leqslant x}\left(1-\frac{n}{x}\right)^2 f(n).$$

The simplification essentially arises from the fact that, if we put (for positive r and x)

$$\begin{aligned} D_r(x) &= \frac{x}{r+1} - \sum_{m\leqslant x}\left(1-\frac{m}{x}\right)^r \qquad (1.2)\\ &= \int_0^x \left(1-\frac{u}{x}\right)^r du - \sum_{m\leqslant x}\left(1-\frac{m}{x}\right)^r\\ &= \int_0^x \left(1-\frac{u}{x}\right)^r d(u-[u])\\ &= \frac{r}{x}\int_0^x \left(1-\frac{u}{x}\right)^{r-1} (u-[u])\, du, \end{aligned}$$

then D_1 has infinite total variation on $(0, +\infty)$, while D_2 has finite total variation there. However, much more than this is true, as we shall prove. Namely, D_2 is actually an increasing function on $(0, +\infty)$. The use of this fact makes the argument very simple indeed and leads to the sharper and more general results quoted above; in particular, it enables us to require only the one-sided boundedness of S, in contrast to the argument of Čudakov and Rodosskiĭ.

In addition, we shall show that D_r is also an increasing function on $(0, +\infty)$ for any r greater than 2. Although there is no advantage in using Riesz means of higher order than the second under the hypotheses of the theorems quoted above, we can use them to obtain interesting results under weaker hypotheses, as the following illustrates.

THEOREM A′. *If* $S(n) \geqslant -K$ *for all* n, *where* $K > 0$, *and if*

$$\sum_{n \leqslant x} \left(1 - \frac{n}{x}\right)^r f(n) \geqslant \tfrac{1}{2} K$$

for some $r \geqslant 2$ *and some positive* x, *then*

$$\liminf_{n \to +\infty} \sum_{m=1}^{n} h(m)\, m^{-1} > 0.$$

Throughout the paper summations over the letters l, m, and n are to be taken over *positive* integers subject to the conditions specified. The letters x and y always denote positive real numbers.

2. Preliminary Lemmas.

LEMMA 1. *For positive real* x *we have*

$$\sum_{m \leqslant x} (x - m)^2 = 2\{\phi_3(x) - \phi_3(x - [x])\},$$

$$\sum_{m \leqslant x} (x - m) = \phi_2(x) - \phi_2(x - [x]),$$

where

$$\phi_3(t) = \frac{t^3}{6} - \frac{t^2}{4} + \frac{t}{12}, \quad \phi_2(t) = \frac{t^2}{2} - \frac{t}{2} + \frac{1}{12}.$$

PROOF. In view of the identity $t^3 = 2\{\phi_3(t+1) - \phi_3(t)\}$, we have

$$\sum_{m=1}^{[x]} (x - m)^2 = \sum_{m=1}^{[x]} 2\{\phi_3(x - m + 1) - \phi_3(x - m)\}$$

$$= 2\{\phi_3(x) - \phi_3(x - [x])\},$$

so that the first assertion is proved. The second assertion is proved in the same way or by differentiation of the first.

LEMMA 2. *If r and x are positive numbers, then*

$$\sum_{n\leqslant x}\left(1-\frac{n}{x}\right)^{r} f(n) = r\int_0^1 (1-u)^{r-1}\left\{\sum_{n\leqslant xu} f(n)\right\} du.$$

PROOF. Both sides are equal to

$$\sum_{n\leqslant x} f(n)\, r\int_{n/x}^{1} (1-u)^{r-1}\, du.$$

LEMMA 3. *If $0 < x \leqslant y$, then*

$$\sum_{n\leqslant x}\left(1-\frac{n}{x}\right) f(n) = \frac{x}{r+1}\sum_{m\leqslant y}\frac{h(m)}{m} -$$

$$-\sum_{m\leqslant y} S(m)\left\{D_r\left(\frac{x}{m}\right) - D_r\left(\frac{x}{m+1}\right)\right\} -$$

$$- S([y])\, D_r\left(\frac{x}{([y]+1)}\right).$$

PROOF. From the definitions (1.1) and (1.2) we have for positive real x

$$\sum_{n\leqslant x}\left(1-\frac{n}{x}\right) f(n) = \sum_{n\leqslant x}\left(1-\frac{n}{x}\right)\sum_{m|n} h(m)$$

$$= \sum_{m\leqslant x} h(m)\sum_{l\leqslant x/m}\left(1-\frac{ml}{x}\right)^r$$

$$= \sum_{m\leqslant x} h(m)\left\{\frac{1}{r+1}\frac{x}{m} - D_r\left(\frac{x}{m}\right)\right\}.$$

Now $D_r(t) = t/(r+1)$ if $0 < t \leqslant 1$ and thus for $y \geqslant x$ we have

$$\sum_{n\leqslant x}\left(1-\frac{n}{x}\right) f(n) = \sum_{m\leqslant y} h(m)\left\{\frac{1}{r+1}\frac{x}{m} - D_r\left(\frac{x}{m}\right)\right\}.$$

The result stated follows from this by partial summation, since $h(1) = S(1)$ and $h(m) = S(m) - S(m-1)$ when $m > 1$.

REMARK. Lemma 3 makes it clear why it would be advantageous to know that D_r is an increasing function. For in that situation we could say that (for $0 < x \leqslant y$)

$$\sum_{n\leqslant x}\left(1-\frac{n}{x}\right)^r f(n)\leqslant \frac{x}{r+1}\sum_{m\leqslant y}\frac{h(m)}{m}-D_r(x)\min_{m\leqslant y}S(m).$$

Otherwise all we could say would be that (for $0<x\leqslant y$)

$$\sum_{n\leqslant x}\left(1-\frac{n}{x}\right)^r f(n)\leqslant \frac{x}{r+1}\sum_{m\leqslant y}\frac{h(m)}{m}+V_r(0,x)\max_{m\leqslant y}|S(m)|,$$

where $V_r(0, x)$ is the total variation of D_r on the interval $(0, x)$.

3. Proofs of the Main Theorems.

LEMMA A. *The function D_2 defined by* (1.2) *is strictly increasing on* $(0, +\infty)$ *and* $D_2(0+)=0$, $D_2(+\infty)=\frac{1}{2}$.

PROOF. Lemma 1 gives

$$D_2(x)=\frac{x}{3}-\frac{2}{x^2}\{\phi_3(x)-\phi_3(x-[x])\}$$

$$=\frac{1}{2}-\frac{1}{6x}+\frac{2}{x^2}\phi_3(x-[x]).$$

The definition of D_2 shows that it has a continuous derivative on $(0,+\infty)$. Hence, since $\phi_3'(t)=\phi_2(t)$, we have

$$D_2'(x)=\frac{1}{6x^2}+\frac{2}{x^2}\phi_2(x-[x])-\frac{4}{x^3}\phi_3(x-[x])$$

for all positive values of x. Now for real t

$$\phi_2(t)=\frac{1}{2}t^2-\frac{1}{2}t+\frac{1}{12}\geqslant\phi_2\left(\frac{1}{2}\right)=-\frac{1}{24},$$

while for $0\leqslant t\leqslant 1$ we have

$$|\phi_3(t)|=\left|\frac{1}{12}(2t-1)(t-1)t\right|\leqslant\frac{1}{12}t(1-t)\leqslant\frac{1}{48}.$$

Hence

$$D_2'(x)\geqslant\frac{1}{6x^2}-\frac{1}{12x^2}-\frac{1}{12x^3}=\frac{1}{12x^2}\left(1-\frac{1}{x}\right)>0$$

if $x>1$. On the other hand $D_2(x)=x/3$ if $0<x\leqslant 1$. Hence D_2 is strictly increasing throughout $(0,+\infty)$. The other assertions of the lemma are immediate.

LEMMA B. *If* $0 < x \leqslant y$, *then*

$$\sum_{m\leqslant y} \frac{h(m)}{m} \geqslant \frac{3}{x}\left(\sum_{n\leqslant x} \left(1-\frac{n}{x}\right)^2 f(n) + \tfrac{1}{2}\min\{0, S(1), S(2), \ldots, S([y])\}\right).$$

PROOF. Lemma A, Lemma 3, and the remark after Lemma 3.

THEOREM A. *If* $S(n) \geqslant -K$ *for all* n, *if* $f(n) \geqslant 0$ *for all* n, *and if* $\sum\limits_{n=1}^{\infty} f(n) > \tfrac{1}{2}K$, *then*

$$\liminf_{n\to+\infty} \sum_{m=1}^{n} h(m)\,m^{-1} > 0.$$

PROOF. We may suppose that $K \geqslant 0$, since, in view of the identity

$$\sum_{m=1}^{n} h(m)\,m^{-1} = \sum_{m=1}^{n-1} S(m)\,\{m^{-1}-(m+1)^{-1}\} + S(n)\,n^{-1}, \qquad (3.1)$$

the assertion of the theorem is trivial (and uninteresting) when K is negative. By the last two assumptions of the theorem there is a positive number x_0 such that

$$\sum_{n\leqslant x_0} \left(1-\frac{n}{x_0}\right)^2 f(n) = \tfrac{1}{2}K + c,$$

where $c > 0$. Then by Lemma B

$$\sum_{m\leqslant y} \frac{h(m)}{m} \geqslant \frac{3c}{x_0}$$

for all $y \geqslant x_0$.

REMARK. The last two assumptions of Theorem A could of course be replaced by the assumption that

$$\sum_{n\leqslant x} \left(1-\frac{n}{x}\right)^2 f(n) > \tfrac{1}{2}K$$

for some positive number x. However a more general result of this nature is given in Theorem A′. A similar remark could be made about Theorem B.

THEOREM B. *If* $S(n) \geqslant -K$ *for all* n, *where* $K \geqslant 0$, *if* $f(n) \geqslant 0$ *for all* n, *and if* $f(n) \geqslant 1$ *whenever* n *is a square, then*

$$\sum_{m=1}^{n} \frac{h(m)}{m} > \frac{32}{75K+150}, \textit{ provided } n \geqslant \frac{225}{64}(K+2)^2.$$

PROOF. The last two assumptions of the theorem imply that

$$\sum_{n \leqslant x} f(n) > x^{\frac{1}{2}} - 1$$

for all positive x. Hence by Lemma 2

$$\sum_{n \leqslant x} \left(1 - \frac{n}{x}\right)^2 f(n) > 2 \int_0^1 (1-u)\,(x^{\frac{1}{2}} u^{\frac{1}{2}} - 1)\, du = \frac{8}{15} x^{\frac{1}{2}} - 1$$

for all positive x. Accordingly, in view of Lemma B,

$$\sum_{m \leqslant y} \frac{h(m)}{m} > \frac{3}{x} \left\{ \frac{8}{15} x^{\frac{1}{2}} - 1 - \tfrac{1}{2} K \right\}$$

if $0 < x \leqslant y$. The right-hand side has its maximum value when

$$\frac{8}{15} x^{\frac{1}{2}} = 2\,(1 + \tfrac{1}{2} K)$$

or $x^{\frac{1}{2}} = \frac{15}{8}(K+2)$. Using this value for x we find that

$$\sum_{m \leqslant y} \frac{h(m)}{m} > \frac{3}{\{(15/8)\,(K+2)\}^2} (1 + \tfrac{1}{2} K) = \frac{32}{75(K+2)},$$

provided $y \geqslant \frac{225}{64}(K+2)^2$.

4. Results obtainable with Higher-Order Riesz Means.

LEMMA A′. *If* $r \geqslant 2$, *the function* D_r *defined by* (1.2) *is strictly increasing on* $(0, +\infty)$ *and* $D_r\,(0+) = 0$, $D_r\,(+\infty) = \frac{1}{2}$.

PROOF. If r and s are positive, we have by (1.2)

$$\int_0^x (x-t)^{s-1}\, t^r\, D_r(t)\, dt = \int_0^x (x-t)^{s-1}\, r \left\{ \int_0^t (t-u)^{r-1} (u - [u])\, du \right\} dt$$

$$= r \int_0^x (u - [u]) \left\{ \int_u^x (x-t)^{s-1} (t-u)^{r-1}\, dt \right\} du$$

$$= \frac{r\,\Gamma(r)\,\Gamma(s)}{\Gamma(r+s)} \int_0^x (u-[u])\,(x-u)^{r+s-1}\,du$$

$$= \frac{\Gamma(r+1)\,\Gamma(s)}{\Gamma(r+s+1)}\, x^{r+s}\, D_{r+s}(x),$$

so that

$$D_{r+s}(x) = \frac{\Gamma(r+s+1)}{\Gamma(r+1)\,\Gamma(s)} \int_0^1 (1-t)^{s-1}\, t^r\, D_r(xt)\, dt.$$

Thus if the conclusions of the theorem are true for some particular value of r, they are true for any larger value. Hence Lemma A′ follows from Lemma A.

LEMMA B′. *If $r \geqslant 2$ and $0 < x \leqslant y$, then*

$$\sum_{m\leqslant y} \frac{h(m)}{m} \geqslant \frac{r+1}{x} \left\{ \sum_{n\leqslant x} \left(1-\frac{n}{x}\right)^r f(n) + D_r(x)\cdot \min_{m\leqslant y} S(m) \right\}.$$

PROOF. Lemma A′, Lemma 3, and the remark after Lemma 3.

THEOREM A′. *If $S(n) \geqslant -K$ for all n, where $K > 0$, and if*

$$\sum_{n\leqslant x} \left(1-\frac{n}{x}\right)^r f(n) \geqslant \tfrac{1}{2}K$$

for some $r \geqslant 2$ and some positive x, then

$$\liminf_{n\to+\infty} \sum_{m=1}^{n} h(m)\, m^{-1} > 0.$$

PROOF. By Lemma B′

$$\sum_{m\leqslant y} \frac{h(m)}{m} \geqslant \frac{r+1}{x} \{\tfrac{1}{2}K - D_r(x)\cdot K\},$$

provided $y \geqslant x$. But $D_r(x) < \frac{1}{2}$ by Lemma A′, and so the result follows.

REMARK. In case S is bounded on both sides, our arguments imply that, if $r \geqslant 2$, then

$$\left|\sum_{n\leqslant x}\left(1-\frac{n}{x}\right)^r f(n)-\frac{x}{r+1}\sum_{m=1}^{\infty}\frac{h(m)}{m}\right|$$
$$\leqslant D_r(x).\sup|S(m)|<\tfrac{1}{2}\sup|S(m)|$$

for any positive x. Thus the content of Theorem A′ in this case is that

$$\sum_{n\leqslant x}\left(1-\frac{n}{x}\right)^r f(n)$$

can never take a value as large as $\frac{1}{2}\sup|S(m)|$ unless it is equal to $C\,x+O(1)$ on $(0,+\infty)$, where C is a positive constant.

THEOREM B′. *If $S(n)\geqslant -K$ for all n, where $K\geqslant 0$, and if there is a number r with $r\geqslant 2$ such that the inequality*

$$\sum_{n\leqslant x}\left(1-\frac{n}{x}\right)^r f(n)\geqslant B\,x^{\frac{1}{2}}-C$$

holds for all positive x, where B and C are positive constants, then

$$\liminf_{n\to+\infty}\sum_{m=1}^{\infty}\frac{h(m)}{m}\geqslant\frac{(r+1)\,B^2}{2K+4C}.$$

PROOF. Similar to the proof of Theorem B.

REMARK. Needless to say, other types of inequalities for the rth order Riesz mean could be used as assumptions in place of the one occurring in Theorem B′. On the other hand, it is easily verified, by using Lemma 2 and Theorem B′, that we would get a poorer numerical result in Theorem B if, in giving its proof, we were to work with a Riesz mean of higher order than the second.

5. The First-Order Riesz Mean. To show the disadvantage of using the first-order Riesz mean for the purposes of the present paper, we now give the analogue of Lemmas A and A′ for this case and the weaker theorems that follow from them. The rather unsatisfactory nature of these results may serve to explain why the use of the first-order Riesz mean for the special case of real non-principal characters χ provides a somewhat inelegant proof of the non-vanishing of $\Sigma\chi(m)\,m^{-1}$ [3, 6, 8, 11, 15].

LEMMA A″. *The function* D_1 *defined by* (1.2) *is not monotonic on* $(0, +\infty)$. *If* $x \geqslant 1$, *the total variation* $V_1(0, x)$ *of* D_1 *on the interval* $(0, x)$ *lies between* $\frac{1}{4}\log x + \frac{1}{4}$ *and* $\frac{1}{4}\log x + \frac{3}{4}$.

PROOF. We have for positive x, by (1.2) and lemma 1,

$$D_1(x) = \tfrac{1}{2} + \frac{1}{2x}(x - [x])(x - [x] - 1).$$

Thus if $0 < x < 1$, $D_1(x) = \frac{1}{2}x$, while if n is a positive integer and $n < x < n + 1$, we have

$$D_1(x) = \tfrac{1}{2}x - n + \frac{n(n+1)}{2x}$$

and

$$D_1{}'(x) = \tfrac{1}{2} - \frac{n(n+1)}{2x^2}$$

Thus $D_1(x)$ decreases when x runs from n to $n^{\frac{1}{2}}(n+1)^{\frac{1}{2}}$ and increases when x runs from $n^{\frac{1}{2}}(n+1)^{\frac{1}{2}}$ to $n+1$. Since $D_1(n) = D_1(n+1) = \frac{1}{2}$ and

$$D_1(n^{\frac{1}{2}}(n+1)^{\frac{1}{2}}) = n^{\frac{1}{2}}(n+1)^{\frac{1}{2}} - n = \tfrac{1}{2} - \frac{1}{2\{n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}}\}^2},$$

we obtain for $x \geqslant 1$

$$\begin{aligned} V_1(0, x) &\leqslant \tfrac{1}{2} + \sum_{n<x} \frac{1}{\{n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}}\}^2} \\ &\leqslant \tfrac{1}{2} + \tfrac{1}{4}\sum_{n<x} \frac{1}{n} \\ &< \tfrac{3}{4} + \tfrac{1}{4}\log x, \end{aligned}$$

and

$$\begin{aligned} V_1(0, x) &\geqslant \tfrac{1}{2} + \sum_{n\leqslant x-1} \frac{1}{\{n^{\frac{1}{2}} + (n+1)^{\frac{1}{2}}\}^2} \\ &\geqslant \tfrac{1}{2} + \tfrac{1}{4}\sum_{n\leqslant x-1} \frac{1}{n+1} \\ &> \tfrac{1}{4} + \tfrac{1}{4}\log x. \end{aligned}$$

LEMMA B″. *If* $1 \leqslant x \leqslant y$, *then*

$$\sum_{m\leqslant y} \frac{h(m)}{m} \geqslant \frac{2}{x}\left\{\sum_{n\leqslant x}\left(1 - \frac{n}{x}\right)f(n) - \left(\tfrac{3}{4} + \tfrac{1}{4}\log x\right)\max_{m\leqslant y}|S(m)|\right\}.$$

PROOF. Lemma A″, Lemma 3, and the remark after Lemma 3.

THEOREM A″. *If* $|S(n)| \leqslant M$ *for all* n *and if*

$$\sum_{n \leqslant x} \left(1 - \frac{n}{x}\right) f(n) > M(\tfrac{3}{4} + \tfrac{1}{4} \log x)$$

for some positive real number x *greater than* 1, *then*

$$\sum_{m=1}^{\infty} h(m)\, m^{-1} > 0.$$

PROOF. Clear. Note that the series $\Sigma\, h(m)\, m^{-1}$ converges under the present assumptions, in view of the identity (3.1).

COROLLARY. *If* $|S(n)| \leqslant M$ *for all* n *and if*

$$\sum_{n \leqslant x} f(n) > M(c + \tfrac{1}{4} \log x)$$

for all sufficiently large positive real x, *where* $c > 1$, *then*

$$\sum_{m=1}^{\infty} h(m)\, m^{-1} > 0.$$

PROOF. We cannot have $M = 0$, for then the functions S, h, and f would all be identically zero. Thus the result follows from Lemma 2 (with $r = 1$) and the preceding theorem.

THEOREM B″. *If* $|S(n)| \leqslant M$ *for all* n, *if* $f(n) \geqslant 0$ *for all* n, *and if* $f(n) \geqslant 1$ *whenever* n *is a square, then*

$$\sum_{m=1}^{\infty} \frac{h(m)}{m} > \frac{1}{3M \log M},$$

provided $M > 10^9$.

PROOF. By Lemma 2 (with $r = 1$)

$$\sum_{n \leqslant x} \left(1 - \frac{n}{x}\right) f(n) > \int_0^1 (x^{\frac{1}{2}} u^{\frac{1}{2}} - 1)\, du = \tfrac{2}{3} x^{\frac{1}{2}} - 1$$

for all positive x. Accordingly, in view of Lemma B″,

$$\sum_{m=1}^{\infty}\frac{h(m)}{m} > \frac{2}{x}\left\{\tfrac{2}{3}x^{\frac{1}{2}} - 1 - (\tfrac{3}{4}+\tfrac{1}{4}\log x)\,M\right\}$$

for any $x \geqslant 1$. Taking $x = ((3/2)\,M\log M)^2$, we readily obtain the result of the theorem.

Needless to say, the results of this section are of interest primarily for their didactic value, since the results of Sections 3 and 4 are stronger in general. Thus the corollary to Theorem A″ is included in Theorem A′, and Theorem B″ is included in Theorem B.

6. Concluding remarks. If χ is a real non-principal character modulo k, it is known [2, 12] that $\Sigma\chi(m)\,m^{-1} > k^{-\epsilon}$ for k sufficiently large and that

$$\sum \chi(m)\,m^{-1} \geqslant k^{-1/2}\log\tfrac{1}{2}(3+\sqrt{5}) > \frac{20}{21}\,k^{-1/2} \tag{6.1}$$

for all k. Our Theorem B is too shallow to give the first of these results, but comes close to giving (6.1) if we use Pólya's estimate [16]

$$\sum_{m=1}^{n}\chi(m) = O(k^{\frac{1}{2}}\log k),$$

which is a consequence of the theory of Gaussian sums. More specifically, it is known [10, 17] that

$$\left|\sum_{m=1}^{n}\chi(m)\right| < \min(\tfrac{1}{2}k,\ (5/3)\ k^{\frac{1}{2}}\log k).$$

Using this in Theorem B, we find that, if χ is a real non-principal character modulo k, then

$$\sum_{m=1}^{\infty}\frac{\chi(m)}{m} > \frac{1}{4k^{\frac{1}{2}}\log k}\,.$$

This is inferior to (6.1) by only a logarithmic factor. Note that Theorem B″ would give a result which is worse by still another logarithmic factor.

The usefulness of the Riesz means of the second and higher orders for our purposes stems from the fact that, if $r \geqslant 2$, the function D_r defined by (1.2) increases from 0 to $\frac{1}{2}$ on $(0, +\infty)$. It is natural to

inquire about the behaviour of D_r when $1 < r < 2$. Certainly D_r is not monotone if $r \leqslant 1.4$, since then

$$D_r(4/3) - D_r(1) = \frac{1}{3(r+1)} - \frac{1}{4^r} = \frac{1}{r+1}\left(\tfrac{1}{3} - \frac{r+1}{4^r}\right) < 0.$$

However, it is possible that D_r has finite total variation on $(0, +\infty)$ if $1 < r < 2$, although we cannot prove this.

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University of Illinois
Urbana, Illinois, USA.

EXISTENCE CONDITIONS FOR AN INTEGRAL OF R. E. LANE

By R. C. BZOCH

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R. E. LANE [1] has defined an integral $L\int_a^b fdg$ which is an extension of the Stieltjes mean σ integral $M\int_a^b fdg$ defined by H. L. Smith [3]. In this paper, using the notation and terminology of [1], we establish a necessary and sufficient condition for the existence of $L\int_a^b fdg$ when f is bounded on $[a, b]$ and g is of bounded variation on $[a, b]$. Our proof for the necessity of the condition uses the existence theorem for the Stieltjes mean σ integral given by P. Porcelli [2]. Our condition is contained in the following:

THEOREM 1. *If f is a bounded function on $[a, b]$ and g is of bounded variation on $[a, b]$, then a necessary and sufficient condition in order that $L\int_a^b fdg$ exist is that there exists a function u such that*

(1) $M\int_a^b udg$ *exists,*

(2) $u = f$ *on a subset G of $[a, b]$ such that each of a and b is in G,*

(3) *if $H = [a, b] - G$, then (g, f, H) is a singular graph, and*

(4) *if x is in G and g is not continuous at x, then x is not an exceptional number for f and g in $[a, b]$.*

We shall prove the sufficiency part of Theorem 1 first. In order to do this, we need the following lemma and definitions.

LEMMA 1. *If f is a bounded function on $[a,b]$, g of bounded variation on $[a, b]$, and p a number in $[a, b]$ such that g is continuous at p, then p is not an exceptional number for f and g in $[a, b]$.*

The proof of Lemma 1 follows easily from Definition 2.2 of [1] and Theorem 1 of [2].

DEFINITION 1. *If each of f and g is a function on* $[a, b]$, $[a', b']$ *a subinterval of* $[a, b]$, *and D is either a subdivision of* $[a, b]$ *containing the points* a' *and* b' *or a subdivision of* $[a', b']$, *then* $S_D(f, g, a', b')$ *will denote the mean sum of f with respect to g taken from* a' *to* b'.

DEFINITION 2. *If G is a subset of* $[a, b]$, *then the statement that D is a G-subdivision of* $[a, b]$, *means that D is a subdivision of* $[a, b]$ *and if x is in D, then x is in G: if D' is a subdivision of* $[a, b]$, *then the statement that E is a G-refinement of D' means that E is a refinement of D', and if x is in E and x is not in D', then x is in G.*

PROOF OF THE SUFFICIENCY. We shall assume that G is a proper subset of $[a, b]$ and show that it is a summability set for f and g on $[a, b]$; i.e. G satisfies conditions (i), (ii), and (iii) of Definition 2.3 of [1]. To this end, we note that (i) is implied by Lemma 1 and (4) of the hypothesis, and (iii) is implied by (3) of the hypothesis.

Since $M\int_a^b udg$ exists, there exists a subdivision D' of $[a, b]$ such that if each of E and F is a refinement of D', then $|S_E(u,g)-S_F(u, g)| < \xi/2$. Let the points of D' be $a = x_0 < x_1 \dots < x_{n+1} = b$. If each point of D' is in G, then we can take $D = D'$ and condition (ii) is satisfied. If D' is not a G-subdivision of $[a, b]$, let $y_1 < y_2 < \dots < y_p$, $p \leqslant n$, be the terms of D' which are also numbers in H.

Since (g, f, H) is a singular graph, there exists a finite collection $\{[a_i, b_i]\}_{i=1}^r$ of non-overlapping subintervals of $[a, b]$ such that for each $i \leqslant r$, (1) a_i and b_i are in G, (2) $a_i < y_i < b_i$, and (3) if A_i and B_i are subdivisions of $[a_i, b_i]$, then $|S_{A_i}(f, g, a_i, b_i) - S_{B_i}(f, g, a_i, b_i)| < \xi/2r$. Let D be a subdivision of $[a, b]$ consisting of the terms of D' that are also numbers in G, and the numbers a_i and b_i, $i = 1, 2, \dots, r$. Let E be a G-refinement of D, and for each $i \leqslant r$, let D_i be the subdivision of $[a_i, b_i]$ consisting of the numbers a_i and b_i and the terms of D' between a_i and b_i. Let D'' be a subdivision of $[a, b]$ consisting of the terms of D from a to a_1, b_1 to a_2, b_2 to $a_3, \dots, b_r$ to b, and the terms of D_i between a_i and b_i for $i = 1, 2, \dots, r$. Let E'' be the subdivision of

$[a, b]$ consisting of the terms of E from a to a_1, b_1 to $a_2, \ldots, b_r$ to b, and the terms of D_i between a_i and b_i for $i=1, 2, \ldots, r$. We have, therefore, $S_{D''}(u, g) = S_D(f, g, a, a_1) + \sum_{i=1}^{r} S_{D_i}(u, g, a_i, b_i) + \sum_{i=1}^{r-1} S_D(f, g, b_i, a_{i+1}) + S_D(f, g, b_r, b)$ and $S_{E''}(u, g) = S_E(f, g, a, a_1) + \sum_{i=1}^{r} S_D(u, g, a_i, b_i) + \sum_{i=1}^{r-1} S_E(f, g, b_i, a_{i+1}) + S_E(f, g, b_r, b)$. Consequently,

$$|S_D(f, g) - S_E(f, g)| \leqslant |S_{D''}(u, g) - S_{E''}(u, g)| + \sum_{i=1}^{r} |S_D(f, g, a_i, b_i) - S_E(f, g, a_i, b_i)| < \xi.$$

This completes the proof of the sufficiency of Theorem 1.

In order to establish the necessity, we shall use the following lemmas and definitions.

Lemma 2. *If each of f and g is a function on $[a, b]$, G a summability set for f and g on $[a, b]$, and $\{P_i\}_{i=1}^n$ is a finite set of numbers in $[a, b]$ but not in G, then the set G' consisting of the numbers $\{P_i\}_{i=1}^n$ and the numbers in G is a summability set for f and g on $[a, b]$.*

Proof of Lemma 2. Suppose $n = 1$ and $\xi > 0$. There exists a G-subdivision D of $[a, b]$ such that if each of E and F is a G-refinement of D, then $|S_E(f, g) - S_F(f, g)| < \xi/3$. There exist numbers c and d in $[a, b]$ such that no term of D is in the segment (c, d), P_1 is in the segment (c, d), and if H and K are subdivisions of the interval $[c, d]$, then $|S_H(f, g, c, d) - S_K(f, g, c, d)| < \xi/3$ and each of c and d is a number in G. Let D' be the subdivision of $[a, b]$ consisting of the terms in D and the numbers c, d, and P_1. Hence, if each of E' and F' is a G-refinement of D' (and, consequently a G'-refinement of D'), then $|S_{E'}(f, g) - S_{F'}(f, g)| < \xi$. That is G' is a summability set for f and g on $[a, b]$. The general case of Lemma 2 now follows by mathematical induction.

Definition 3. *If f is a bounded function on $[a, b]$, G a subset of $[a, b]$, and $k > 0$, then*

(i) *$M_G(f, k^+)$ denotes a subset of $[a, b]$ such that x is in $M_G(f, k^+)$ if, and only if, $x < b$ and for each number y in the segment (x, b)*

there exist numbers c *and* d *in* G *and in the segment* (x, y) *such that* $|f(c)-f(d)| \geqslant k$;

(ii) $M_G(f, k^-)$ *denotes a subset of* $[a, b]$ *such that* x *is in* $M_G(f, k^-)$ *if, and only if,* $a < x$ *and for each number* y *in the segment* (a, x) *there exist numbers* c *and* d *in* G *and in the segment* (y, x) *such that* $|f(c)-f(d)| \geqslant k$;

(iii) $M_G(f, k)$ *denotes the logical sum of* $M_G(f, k^+)$ *and* $M_G(f, k^-)$. *If* $G = [a, b]$, *then we denote* $M_G(f, k)$ *by* $M(f, k)$: (cf. Def. 1.1 of [2]).

DEFINITION 4. *If* f *is a bounded function on* $[a, b]$, g *a nondecreasing function on* $[a, b]$, G *a subset of* $[a, b]$, *and* $k > 0$, *then the statement that* $M_G(f, k)$ *has directed* g*-content* $C_g\, M_G(f, k)$ *means that* $C_g\, M_G(f, k)$ *is the largest nonnegative number* S *such that if* $\{[a_i, b_i]\}_{i=1}^n$ *is a finite collection of pairwise mutually exclusive subintervals of* $[a, b]$ *satisfying*

(i) a_i *is not in* $M_G(f, k^-)$ *and* b_i *is not in* $M_G(f, k^+)$ *for* $i = 1, \ldots, n$, *and* (ii) $\{[a_i, b_i]\}_{i=1}^n$ *covers* $M_G(f, k)$, *then* $\sum_{i=1}^{n} [g(b_i)-g(a_i)] \geqslant S$.

LEMMA 3. *If* f *is a bounded function on* $[a, b]$, g *a function of bounded variation on* $[a, b]$, G *a summability set for* f *and* g *on* $[a, b]$, *and* $k > 0$, *then* $C_{g'}\, M_G(f, k) = 0$, *where* g' *is the variation function of* g.

PROOF OF LEMMA 3. Let $\xi > 0$. There exists a subdivision D' of $[a, b]$ such that if E', denoted by $a = x_0 < x_1 < \ldots < x_{n+1} = b$, is a refinement of D', then $\sum_{i \in R} |g(x_{i+1})-g(x_i)| > \{\sum_{i \in R} g'(x_{i+1})-g'(x_i)\} - \xi/2$, where R is any subset of the integers $0, 1, \ldots, n$.

If there exist terms of D' that are not in G, then by Lemma 2 the set G' consisting of G and these terms is a summability set for f and g in $[a, b]$ and, moreover, $C_{g'}\, M_{G'}(f, k) = 0$ if, and only if, $C_{g'} M_G(f, k) = 0$. Consequently, we shall assume that D' is a G-subdivision of $[a, b]$. Also, since G is a summability set for f and g in $[a, b]$, there exists a G-refinement, say D, of D' such that if each of E and F is G-refinement of D, then $|S_E(f, g)-S_F(f, g)| < k\xi/8$. We shall denote the subdivision D by $a = x_0 < x_1 \ldots < x_q = b$. Let T denote

a collection of subintervals of $[a, b]$ such that $[c, d]$ is in T if, and only if, (1) there exists a non-negative integer $i \leqslant q-1$, such that $[c, d] = [x_i, x_{i+1}]$, and (2) either the left open interval $(c, d]$ contains a point of $M_G(f, k^-)$ or the right open interval $[c, d)$ contains a point of $M_G(f, k^+)$. If m denotes the number of intervals in T, then $m \leqslant q$. Let R be a subset of the numbers $0, 1, \ldots, q-1$, such that i is in R if, and only if, $[x_i, x_{i+1}]$ is in T.

For each integer p in R, there exists a number z_p in $[x_p, x_{p+1}]$ such that z_p is in $M_G(f, k^-)$ or in $M_G(f, k^+)$. If z_p is in $M_G(f, k^+)$ then (in view of the definitions of T, $M_G(f, k^+)$, and the fact that g is of bounded variation) there exist numbers s_p and r_p in G such that, (1) s_p and r_p are in the segment (z_p, x_{p+1}), (2) $|f(r_p) - (fs_p)| \geqslant k$, (3) $|g(r_p) - g(s_p)| \leqslant (\xi k m N)/8$, where N is an upper bound of $|f|$ on $[a, b]$, and (4) the numbers $[g(x_{p+1}) - g(x_p)]$ and $[f(r_p) - f(s_p)]$ have the same algebraic sign. If z_p is in $M_G(f, k^-)$, then there exist numbers s_p and r_p in G and in the segment (x_p, z_p) satisfying conditions 2, 3, and 4.

If E is the G-refinement of D consisting of just the terms of D and the numbers r_p, p in R, and F is the G-refinement of D consisting of just the terms of D and the numbers s_p, p in R, then

$$k\xi/8 > |S_E(f, g) - S_F(f, g)| \geqslant \sum_{p \xi R} (2^{-1}[f(r_p) - f(s_p)]\,[g(x_{p+1}) - g(x_p)] - \\ - 2^{-1}[f(x_{p+1}) - f(x_p)]\,[g(r_p) - g(s_p)]),$$

or $\sum_{p \xi R} |g(x_{p+1}) - g(x_p)| < \xi/2$, so that, since D is a refinement of D', $\sum_{p \xi R} [g(x_{p+1}) - g(x_p)] < \xi$.

If for some p in R, x_{p+1} is in $M_G(f, k^+)$, then we can replace the two intervals $[x_p, x_{p+1}]$ and $[x_{p+1}, x_{p+2}]$ by the interval $[x_p, x_{p+2}]$ and, since g' is an additive interval function, the above inequality on g' will remain valid. A similar argument holds if for some p in R x_p is in $M_G(f, k^-)$. It follows, therefore, that we can obtain from T a finite collection $\{[a_i, b_i]\}_{i=1}^{\gamma}$ covering $M_G(f, k)$ and satisfying (i) and (ii) of Definition 4. Since $\sum_{i=1}^{\gamma} [g'(b_i) - g'(a_i)] < \xi$, it follows that $C_{g'} M_G(f, k) = 0$.

This completes the proof of Lemma 3.

DEFINITION 5. *Let f be a bounded function on $[a, b]$, g a function of bounded variation on $[a, b]$, G a summability set for f and g in $[a, b]$, and $H = [a, b] - G$. We define a function u on $[a, b]$ as follows:*

(i) *if y is in G, then $u(y) = f(y)$,*

(ii) *if y is in H and y is a limit point of G from the right, then $u(y) = \limsup_{x=y+, x\in G} f(x)$; if y is in H and y is a limit point of G only from the left, then $u(y) = \limsup_{x=y-, x\in G} f(x)$, and*

(iii) *if y is in H and y is not a limit point of G, then let c denote the* LUB *of all number c' in G such that $c' < y$ and let d denote the* GLB *of all numbers d' in G such that $y < d'$: each number x in the segment is in H and is not a limit point of G, for each such number we define $u(x) = (d-c)^{-1}[u(d) - u(c)]\, x + (d-c)^{-1}[du(c) - cu(d)]$.*

LEMMA 4. *If f is a bounded function on $[a, b]$, g a function of bounded variation on $[a, b]$, G a summability set for f and g in $[a, b]$, u the function defined as in Definition 5, $k > 0$, and p a number in $[a, b]$ such that p is not in $M_G(f, k/16)$, then p is not in $M(u, k)$.*

PROOF OF LEMMA 4. If p is not a limit point of H, there exist numbers $y' < p$ and $y'' > p$ such that if x is a number in either (y', p) or (p, y''), then x is in G and $f(x) = u(x)$ so that p is not in $M(u, k/16)$ and, consequently, p is not in $M(u, k)$.

Let us suppose now that p is a limit point of H, p is in, say, $M(u, k^+)$, and p is not a limit point of G from the right. From the definition of $M(u, k^+)$, $p \neq b$ so that there exists a number q such that $p < q \leqslant b$ and if x is in the segment (p, q), then x is in H and, from (iii) of Definition 5, u is a linear function in the interval $[p, q]$. Therefore, u is continuous from the right at p and p is not in $M(u, k^+)$. This contradiction shows that p must be a limit point of G from the right.

Since p is a right sided limit point G, there exists a number q such that $p < q \leqslant b$, and that if each of z and z' is a number in

G and in (p, q), then $|f(z) - f(z')| = |u(z) - u(z')| < k/16$. Since p is in $M(u, k^+)$, there exist numbers t and t' in (p, q) such that $|u(t) - u(t')| \geqslant k$ and, moreover, we can always select t and t' so that there are numbers in G and in (p, q) to the left and to the right of each of t and t'. We shall now consider cases in which the second case of the four will be broken down into four subcases.

(1) Each of t and t' is in G. This gives a contradiction immediately.

(2) Suppose t is in G and t' is not in G. If t' is a limit point of G then, by (ii) of Definition 5, $|u(t) - u(t')| \leqslant k$, which is a contradiction. Hence, t' is not a limit point of G and, in accordance with (iii) of Definition 5, there exist numbers c and d in (p, q) such that $c < t' < d$ and there exist numbers γ and π such that if x is in $[c, d]$ then $u(x) = \gamma x + \pi$.

(2_1) If each of c and d is in G, then $|\gamma|\,|c - d| = |u(c) - u(d)| < k/16$, so that $|u(c) - u(t')| = |\gamma(c - t')| \leqslant |\gamma|\,|c - d| < k/16$. This implies that $|u(t) - u(t')| < k/8$, which is a contradiction.

(2_2) Suppose that c is not in G and d is in G. We can assume that c is a limit point of G from the left, hence there exists a number x' in (p, q) and in G and to the left of c such that $|u(c) - u(x')| < k/16$. This implies $|u(t') - u(d)| < |\gamma|\,|c - d| = |u(c) - u(d)| \leqslant |u(c) - u(x')| + |u(x') - u(d)| < k/8$, so that $|u(t) - u(t')| < 3k/16$ which is a contradiction.

(2_3) Suppose c is in G and d is not in G. This case is essentially the same as 2_2.

(2_4) Suppose each of c and d is not in G. In this case there exist numbers x' and x'' in (p, q) and in G such that $x' < c$, $d < x''$, $|u(x') - u(c)| < k/16$, and $|u(x'') - u(d)| < k/16$. Now, an argument similar to the one used in 2_2 gives us $|u(t) - u(t')| < 5k/16$. This contradiction completes case 2.

(3) Suppose t' is in G and t is not in G. This is the same as case 2 with t and t' interchanged.

(4) Suppose neither t nor t' is in G. Since t is not in G, then there exists a number y in G and in (p, q) such that $|u(y) - u(t)| < k/4$.

Hence, if x is any number in G and in (p, q) we have $|u(t) - u(x)| < 5k/16$, so that $|u(t') - u(x)| \geqslant |u(t') - u(t)| - |u(t) - u(x)| > k - 5k/16 = 11k/16$. This reduces case 4 to case 2 and completes the proof of Lemma 4.

LEMMA 5. *If f is a bounded function on $[a, b]$, g a function of bounded variation on $[a, b]$ G a summability set for f and g in $[a, b]$, $H = [a, b] - G$, $k > 0$, and u defined as in Definition 5, then $C_{g'} M(u, k) = 0$, where g' is the variation function of g.*

PROOF OF LEMMA 5. By Lemma 4, the only numbers in $M(u, k)$ are numbers in $M_G(f, k/16)$. By Lemma 3, $C_{g'} M_G(f, k/16) = 0$, therefore, $C_{g'} M(u, k) = 0$.

PROOF OF THE NECESSITY OF THEOREM 1. Since $L\int_a^b fdg$ exist, there exists a summability set, say G, for f and g in $[a, b]$. Let $H = [a, b] - G$ and u be defined as in Definition 5, so that $u = f$ except, at most, on a set H, where (g, f, H) is a singular graph; i.e. conditions (2) and (3) are satisfied. The existence of $L\int_a^b fdg$ implies that there are no exceptional numbers for f and g in G so that (4) is satisfied. By Lemma 5 and Theorem 1 of [2], $M\int_a^b vdg$ exists so that (1) is satisfied.

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University of Minnesota
Minneapolis, Minnesota

CERTAIN CONVERGENCE THEOREMS CONNECTED WITH A GENERALIZED HANKEL-TRANSFORM

By RAM KUMAR

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1. Agarwal in 1950 introduced a generalized Hankel-transform and obtained certain inversion formulae (1950, 1953) and the fundamental properties (1951) for it. In particular, he proved that, under appropriate conditions, if

$$f(x) = (1/2)^{\lambda} \int_0^{\infty} (xy)^{\lambda+\frac{1}{2}} J_{\lambda}^{\mu}(\tfrac{1}{4} x^2 y^2)\, g(y)\, dy, \tag{1.1}$$

then

$$g(y) = \frac{1}{\mu} 2^{\lambda-2(\lambda+1)/(\mu)+2} \int_0^{\infty} (xy)^{2(\lambda+1)/(\mu)-\lambda-3/2} J^{1/\mu}_{(\lambda+1)/(\mu)-1}(\tfrac{1}{4} x^2 y^2)^{1/\mu} \times$$
$$\times f(x) dx. \tag{1.2}$$

Recently, Stankovic (1955) proved the inversion formula under less restrictive conditions and obtained the general form of functions self-reciprocal under this transform. I have in a series of papers studied the recurrence relations (1954, '55), integral representations (1957), infinite series expansions (1957), connections with other transforms (1956,'57) etc. of an equivalent transform

$$f(x) = \int_0^{\infty} (xy)^{\nu} J_{\lambda}^{\mu}(xy)\, g(y)\, dy, \tag{1.3}$$

where $J_{\lambda}^{\mu}(x)$ is the Bessel-Maitland function defined by

$$J_{\lambda}^{\mu}(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{\underline{|r}\, \Gamma(1+\lambda+\mu r)}, \quad \mu > 0.$$

In view of the interesting nature of this generalized Hankel-transform, I give herewith certain convergence and order theorems for it. We shall discuss the convergence properties of the integral

$$\int_0^\infty (xy)^\nu J_\lambda^\mu(xy)\ d\alpha(y) \tag{1.4}$$

instead of (1.3) where $\alpha(y)$ is a function of the real variable y in the interval $0 \leqslant y < \infty$ and is of bounded variation in $0 \leqslant y \leqslant R$ for every positive R.

When the integral (1.4) converges, it defines a function of x which we shall call the generalized Hankel-Stieltjes transform of $\alpha(y)$ and shall denote it by $f(x)$. If

$$f(x) = \int_0^\infty (xy)^\nu J_\lambda^\mu(xy)\ g(y)\ dy,$$

we shall refer to $f(x)$ as the generalized Hankel-transform of $g(y)$. The term generating function shall be used for $f(x)$ and the determining function both for $\alpha(y)$ and $g(y)$.

2. THEOREM. *If*

$$\underset{0 \leqslant u < \infty}{u.b.} \left| \int_0^u (x_0\, y)^\nu J_\lambda^\mu(x_0\, y)\ d\alpha(y) \right| = M < \infty, \tag{2.1}$$

then the integral

$$\int_0^\infty (xy)^\nu J_\lambda^\mu(xy)\ d\alpha(y) \tag{2.2}$$

converges for $\mu < 1$ *and for every* x *for which* $x > x_0$ *and is equal to*

$$(x/x_0)^\nu \int_0^\infty \psi(xy)\ \beta(y)\ dy, \tag{2.3}$$

where

$$\psi(xy) = \frac{x_0\, J_\lambda^\mu(xy)\, J_{\lambda-1+\mu}^\mu(x_0\, y) - x\, J_\lambda^\mu(x_0\, y)\, J_{\lambda-1+\mu}^\mu(xy)}{\{J_\lambda^\mu(x_0\, y)\}^2} \tag{2.4}$$

and

$$\beta(u) = \int_0^u (x_0\, y)^\nu J_\lambda^\mu(x_0\, y)\ d\alpha(y). \tag{2.5}$$

PROOF. *If* $\beta(u)$ is defined by 2(5), then (Widder, 1946, p. 12)

$$(x_0/x)^\nu \int_0^R (xy)^\nu J_\lambda^\mu(xy)\, d\alpha(y) = \int_0^R \frac{J_\lambda^\mu(xy)}{J_\lambda^\mu(x_0\, y)}\, d\beta(y),$$

$$= \left[\frac{J_\lambda^\mu(xy)}{J_\lambda^\mu(x_0\, y)} \beta(y) \right]_0^R - \int_0^R \frac{d}{dy} \left\{ \frac{J_\lambda^\mu(xy)}{J_\lambda^\mu(x_0\, y)} \right\} \beta(y)\, dy.$$

Now $[J_\lambda^\mu(xy)/J_\lambda^\mu(x_0\, y)]\ \beta(y)$ vanishes at $y = 0$ for $J_\lambda^\mu(y) = O(1)$ when y is small (Wright, 1935) and $\beta(y)$ vanishes at $y = 0$ by virtue of (2.5).

Also $\underset{R\to\infty}{\mathrm{Lt.}}\ [J_\lambda^\mu(xR)/J_\lambda^\mu(x_0\, R)]\ \beta(R) = 0$ when $x > x_0$ and $\mu < 1$ on account of the relation 2(1) and the following asymptotic behaviour of $J_\lambda^\mu(x)$ (Wright, 1935)

$$J_\lambda^\mu(x) = O[x^{-k'(\lambda+\frac{1}{2})} \exp\{(\mu x)^{k'} (\cos \pi k')/\mu k'\}], \tag{2.6}$$

where $k' = 1/(1 + \mu)$ and x is large.

Now

$$\left| \int_0^R \frac{d}{dy} \left\{ \frac{J_\lambda^\mu(xy)}{J_\lambda^\mu(x_0 y)} \right\} \beta(y)\, dy \right| \leqslant M \int_0^R \frac{d}{dy} \left\{ \frac{J_\lambda^\mu(xy)}{J_\lambda^\mu(x_0 y)} \right\} dy,$$

$$= M \left[\frac{J_\lambda^\mu(xy)}{J_\lambda^\mu(x_0 y)} \right]_0^R .$$

From (2.6), it is evident that, under the conditions stated, the right hand side of this inequality is finite, however large R may be. Therefore, by virtue of the relation (Wright, 1933)

$$\frac{d}{dx} J_\lambda^\mu(x) = - J_{\lambda-1+\mu}^\mu(x), \tag{2.7}$$

we have

$$\int_0^\infty (xy)^\nu J_\lambda^\mu(xy)\ d\alpha(y) = (x/x_0)^\nu \int_0^\infty \frac{d}{dy} \left\{ \frac{J_\lambda^\mu(xy)}{J_\lambda^\mu(x_0 y)} \right\} \beta(y) dy,$$

$$= (x/x_0)^\nu \int_0^\infty \psi(x,y)\ \beta(y)\ dy.$$

COROLLARY. *If the integral* (2.2) *converges for* $x = x_0$, *it converges for all* $x > x_0$.

For, if (2.2) *converges for* $x = x_0$, *then* (2.1) *holds and hence* (2.2) *converges for all* $x > x_0$.

3. We now proceed to determine from the convergence properties of the integral the order properties of the corresponding determining function.

THEOREM. *If the integral*

$$\int_0^\infty (xy)^\nu \ J_\lambda^\mu(xy) \ d\alpha(y) \tag{3.1}$$

converges for $\nu = \nu_0$, *then*

$$\alpha(y) = o\left[y^{k'(\lambda+\frac{1}{2})-\nu} \exp[-(\mu y)^{k'} (\cos \pi k')/\mu k']\right], \ k' = 1/(1+\mu), \tag{3.2}$$

for every positive x, $R(\nu_0) \equiv \gamma$ *and* $0 < \mu \leqslant 1$ *with an additional condition* $R\ [2\nu_0 - k'(\lambda + \frac{1}{2})] < 0$ *in case* $\mu = 1$.

PROOF. Let

$$\beta(t) = \int_0^t (xy)^{\nu_0} J_\lambda^\mu(xy) \ d\alpha(y), \ 0 < t < \infty; \tag{3.3}$$

then (Widder, p. 12)

$$\int_0^t d\,\alpha(y) = \int_0^t \frac{d\,\beta(y)}{(xy)^{\nu_0} \ J_\lambda^\mu(xy)},$$

or
$$\alpha(t) - \alpha(0) = \frac{\beta(t)}{(xt)^{\nu_0} \ J_\lambda^\mu(xt)} - \int_0^t \beta(y) \frac{d}{dy} \left[(xy)^{\nu_0} \ J_\lambda^\mu(xy)\right]^{-1} dy,$$

since $\lim\limits_{y\to 0} \ [\beta(y)/(xy)^{\nu_0} \ J_\lambda^\mu(xy)]$ tends to zero by virtue of (3.3) and the asymptotic expansion $J_\lambda^\mu(y) = O(1)$ when $y \to 0$.

Also, by hypothesis, $\beta(\infty)$ exists. Therefore

$$\underset{t\to\infty}{\text{Lt.}}\ [\alpha(t)-\alpha(0)]\,(xt)^{\nu_0}\,J_\lambda^\mu(xt)$$

$$=\beta(\infty)-\underset{t\to\infty}{\text{Lt.}}\ (xt)^{\nu_0}\,J_\lambda^\mu(xt)\int_0^t \beta(y)\,\frac{d}{dy}\,[(xy)^{\nu_0}\,J_\lambda^\mu(xy)]^{-1}\,dy,$$

$$=\underset{t\to\infty}{\text{Lt.}}\ (xt)^{\nu_0}\,J_\lambda^\mu(xt)\int_0^t [\beta(\infty)-\beta(y)]\,\frac{d}{dy}\,[(xy)^{\nu_0}\,J_\lambda^\mu(xy)]^{-1}\,dy,$$

when $0<\mu\leqslant 1$ with an additional condition $R[2\nu_0-k'(\lambda+\frac{1}{2})]<0$ in case $\mu=1$.

Evidently the right hand side vanishes under the conditions of the theorem. Hence

$$\alpha(t)-\alpha(0)=o\,[(xt)^{\gamma}\,J_\lambda^\mu(xt)]^{-1},\quad t\to\infty,$$

or $$\alpha(y)=o[y^{k'(\lambda+\frac{1}{2})-\gamma}\exp\{-(\mu y)^{k'}\,(\cos\pi k')/\mu k'\}]$$

for every positive x and $0<\mu\leqslant 1$ with an additional condition $R[2\nu_0-k'(\lambda+\frac{1}{2})]<0$ in case $\mu=1$.

Hence the proposition.

4. We shall now investigate the relation between the convergence properties of (1.4) and the order properties of the corresponding determining function.

THEOREM. *If*

$$\alpha(y)=O[y^{k'(\lambda+\frac{1}{2})-\gamma}\exp\{-(\mu y)^{k'}\,(\cos\pi k')/\mu k'\}], \tag{4.1}$$

where $k'=1/(1+\mu)$, *for some number* γ, *then the integral*

$$\int_0^\infty (xy)^{\nu}\,J_\lambda^\mu(xy)\,d\alpha(y) \tag{4.2}$$

converges for every $x\geqslant 1$, $R(\nu)>0$ *and* $\mu\leqslant 1$ *with an additional condition* $\gamma>R(\nu)$ *in case* $\mu=1$ *or* $x=1$ *or both.*

PROOF. Since $\alpha(y)$ is of bounded variation in every finite interval, it follows from 4(1) that a constant M exists such that

$$|\alpha(y)| \leqslant M\, y^{k'(\lambda+\frac{1}{2})-\gamma} \exp[-(\mu y)^{k'} (\cos \pi k')/\mu k'],$$

where $0 \leqslant y < \infty$ and $k' = 1/(1+\mu)$. Hence*

$$\int_0^R \frac{d}{dy}[(xy)^\nu J_\lambda^\mu(xy)]\,\alpha(y)\,dy$$

$$\ll M(R)^{k'(\lambda+\frac{1}{2})-\gamma} \exp[-(\mu R)^{k'} (\cos \pi k')/\mu k'] \int_0^R \frac{d}{dy}[(xy)^\nu J_\lambda^\mu(xy)]\,dy,$$

$$= M x^{\nu - k'(\lambda+\frac{1}{2})} R^{\nu-\gamma} \exp[-(1-x^{k'})(\mu R)^{k'} (\cos \pi k')/\mu k'] \text{ for } R(\nu) > 0.$$

Therefore the integral on the left hand side of the inequality converges for $R(\nu) > 0$, $\mu \leqslant 1$ and for every $x \geqslant 1$ with an additional condition $\gamma > R(\nu)$ in case $\mu = 1$ or $x = 1$ or both.

Now

$$\int_0^R (xy)^\nu J_\lambda^\mu(xy)\, d\alpha(y) = (xR)^\nu J_\lambda^\mu(xR)\,\alpha(R) - \int_0^R \frac{d}{dy}[(xy)^\nu J_\lambda^\mu(xy)]\,\alpha(y)\,dy.$$

But $(xR)^\nu J_\lambda^\mu(xR)\,\alpha(R) \to 0$ when $R \to \infty$ for $\mu \leqslant 1$ and for every $x \geqslant 1$, with an additional condition $\gamma > R(\nu)$ in case $\mu = 1$ or $x = 1$ or both.

Hence (4.2) converges under the conditions of the theorem.

Corollary. *If $\alpha(\infty)$ exists and if*

$$\alpha(y) - \alpha(\infty) = O[y^{k'(\lambda+\frac{1}{2})-\gamma} \exp\{-(\mu y)^{k'} (\cos \pi k')/\mu k'\}],$$

where $k' = 1/(1+\mu)$ for some real number γ, then the integral (4.2) converges for every $x \geqslant 1$, $R(\nu) > 0$ and $\mu \leqslant 1$ with an additional condition $\gamma > R(\nu)$ in case $\mu = 1$ or $x = 1$ or both.

This is evident for

$$\int_0^\infty (xy)^\nu J_\lambda^\mu(xy)\, d[\alpha(y) - \alpha(\infty)] = \int_0^\infty (xy)^\nu J_\lambda^\mu(xy)\, d\alpha(y).$$

* The symbol $\ll$ means that the integral on the left is dominated by the one on the right, or that the integrand of the first is in absolute value not greater than that of the second over the whole range of integration.

5. With the help of the result of the theorem of § 3, we can express a generalized Hankel-Stieltjes transform in terms of an ordinary Hankel-transform.

THEOREM. *If the integral*

$$f(x) = \int_0^\infty (xy)^\nu J_\lambda^\mu(xy)\, d\alpha(y)$$

converges for every positive x, $R(\nu) > 0$ *and* $0 < \mu \leqslant 1$ *with an additional condition* $R[2\nu - k'(\lambda + \frac{1}{2})] < 0$ *in case* $\mu = 1$, *then*

$$\int_0^\infty (xy)^\nu J_\lambda^\mu(xy)\, d\,\alpha(y)$$

$$= x \int_0^\infty (xy)^\nu J_{\lambda-1+\mu}^\mu(xy)\,\alpha(y)\, dy + \nu x \int_0^\infty (xy)^{\nu-1} J_\lambda^\mu(xy)\,\alpha(y)\, dy.$$

PROOF. Evidently

$$\int_0^R (xy)^\nu J_\lambda^\mu(xy)\, d\alpha(y) = (xR)^\nu J_\lambda^\mu(xR)\,\alpha(R) - \int_0^R \frac{d}{dy}[(xy)^\nu J_\lambda^\mu(xy)]\,\alpha(y)\, dy,$$

provided $R(\nu) > 0$.

But, by the theorem of § 3, we have

$$(xR)^\nu J_\lambda^\mu(xR)\,\alpha(R) \to 0 \text{ as } R \to \infty,$$

under the conditions of the theorem.

Therefore, on making use of (2.7), we get the required result.

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Roorke University
Roorkee

TWO APPLICATIONS OF A RESULT OF S. CHOWLA

By M. BHASKARAN

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1. In this paper, I give two applications of the following result of Chowla [2] :

Let $\{m, n\}$ denote the number of different pairs of numbers x, y such that $(x, y) = 1$, $1 \leqslant x \leqslant m$, $1 \leqslant y \leqslant n$.

Then

$$\{m, n\} = \frac{6\,mn}{\pi^2} + O(m \log n)$$

if $m \geqslant n$.

2. THEOREM 1. *There is a quadratic non-residue of p between $p^{1/2}/12$ and $p^{1/2}$ for all $p \geqslant p_0$, where p is a rational prime.*

First we shall prove that different pairs x, y satisfying

$$(x, y) = 1, p^{1/2}/12 < x < p^{1/2}, \frac{p^{1/2}}{12} < y < p^{1/2} \tag{2.1}$$

give rise to different values of z modulo p for the congruence

$$x \equiv yz (\text{mod } p).$$

For, if not, let there exist two pairs x_1, y_1 and x_2, y_2 which give rise to the same value z_0 of z mod p.

Then the congruences

$$x_1 \equiv y_1 z_0 \text{ and } x_2 \equiv y_2 z_0 (\text{mod } p) \text{ lead to}$$

$$(x_2 y_1 - x_1 y_2)\, z_0 \equiv 0 (\text{mod } p).$$

Here x_1 being prime to p, z_0 is prime to p. So

$$x_2 y_1 - x_1 y_2 \equiv 0 (\text{mod } p). \tag{2.2}$$

Also

$$x_2 y_1 - x_1 y_2 < [p^{1/2}]\, [p^{1/2}] < p. \tag{2.3}$$

From (2.2) and (2.3) we get

$$x_2 y_1 = x_1 y_2. \tag{2.4}$$

(2.4) is satisfied only when $x_1 = x_2$ and $y_1 = y_2$ because of the condition $(x_r, y_r) = 1$ $(r = 1, 2)$. Hence the pairs taken are the same. This implies (2.1).

PROOF. Now the number of pairs x, y satisfying the conditions of the theorem is equal to

$$\{[p^{1/2}], [p^{1/2}]\} - 2\{[p^{1/2}/12], [p^{1/2}]\} + \{[p^{1/2}/12], [p^{1/2}/12]\}$$

$$= \frac{6p}{\pi^2} + O(p^{1/2}\log p^{1/2}) - \frac{1}{\pi^2}p + \frac{p}{24\pi^2} \text{ (by Chowla's result)}$$

$$> p/2 \text{ for } p \geqslant p_0. \tag{2.5}$$

From (2.1) and (2.5) it follows that z runs through more than $(p-1)/2$ residues.

But there are only $(p-1)/2$ quadratic residues. So there must be at least one z which is a quadratic non-residue of p. But then it follows from the congruence $x \equiv yz$ that one of x and y is also a quadratic non-residue of p. Hence the theorem.

3. THEOREM 2. *The number of ways in which a prime p can be expressed in the form $u_1 v_1 + u_2 v_2$, where $1 \leqslant u_r \leqslant p^\epsilon$, $1 \leqslant v_r \leqslant p^{1-\epsilon}$, $(r = 1, 2)$, $0 < \epsilon < 1$ is equal to*

$$\left(\frac{12}{\pi^2} - 1\right)p + O(p^{\max(1-\epsilon,\epsilon)}\log p).$$

To prove this, we require the following lemmas.

LEMMA 1. *Suppose that p is a prime number, a and b are integers not divisible by p, and r and s are positive integers whose product is greater than p. Then there exist relatively prime integers x and y such that*

$$ax \equiv by \pmod p,\quad 0 < x < r,\quad 0 < |y| < s.$$

This result is well known. Here it is stated in a form suitable for its actual use in the paper. For proof refer ([1], [3] Ex. 8, p. 237, [5], [6] Satz. 28).

Lemma 2. *Suppose that* ϵ *is a given positive number between* 0 *and* 1, p *is a prime number, and* a *is an integer not divisible by* p. *Let* S *be the set of pairs of integers* x, y *such that*

$$1 \leqslant x \leqslant p^{\epsilon},\ 1 \leqslant |y| \leqslant p^{1-\epsilon},\ (x, y) = 1.$$

To each pair x, y *in* S *there corresponds a unique* b *modulo* p, *such that* $ax \equiv by \pmod{p}$.

Two pairs x_1, y_1 *and* x_2, y_2 *give rise to the same value of* b *if and only if* y_1 *and* y_2 *have opposite signs and* $x_2\,|y_1| + x_1\,|y_2| = p$.

Proof. Suppose that for some pair belonging to S, the congruence $ax \equiv by \pmod{p}$ is satisfied for $b = b_1$ and b_2.

Then it easily follows that $b_1 \equiv b_2 \pmod{p}$.

Hence to each pair x, y belonging to S, there corresponds a unique b modulo p such that $ax \equiv by \pmod{p}$.

To prove the second part of the lemma, suppose two different pairs x_1, y_1 and x_2, y_2 give rise to the same value of b modulo p. Then

$$ax_1 \equiv by_1,\ ax_2 \equiv by_2 \pmod{p}.$$

These two congruences lead to $b(x_1 y_2 - x_2 y_1) \equiv 0 \pmod{p}$.

Since

$$b \not\equiv 0 \pmod{p},$$

$$x_1 y_2 - x_2 y_1 \equiv 0 \pmod{p}.$$

If y_1 and y_2 have the same sign $|x_1 y_2 - x_2 y_1| < [p^{\epsilon}]\,[p^{1-\epsilon}] < p$.

Consequently $x_1 y_2 - x_2 y_1 = 0$. Hence, x_r being prime to y_r $(r = 1, 2)$ it follows that

$$x_1 = x_2,\ y_1 = y_2.$$

So in this case the pairs are the same.

If y_1 and y_2 have opposite signs, it is easy to see that

$$|x_1 y_2 - x_2 y_1| = x_1\,|y_2| + x_2\,|y_1|$$

is positive and $\quad < 2[p^{\epsilon}]\,[p^{1-\epsilon}] < 2p$.

Hence $\quad x_1\,|y_2| + x_2\,|y_1| = p$.

This implies that two different pairs give rise to the same value of b (mod p) only if y_1 and y_2 have opposite signs and $x_1 |y_2| + x_2 |y_1| = p$.

Conversely, if y_1 and y_2 have opposite signs and $x_1 |y_2| + x_2 |y_1| = p$, the pairs x_1, y_1 and x_2, y_2 give rise to the same value of b modulo p.

For, $x_1 |y_2| + x_2 |y_1| = p$ with $y_1 y_2 < 0$ implies that

$$x_1 y_2 \equiv x_2 y_1 \pmod{p}.$$

This, together with $ax_1 \equiv by_1$, implies that $ax_1 y_2 \equiv by_1 y_2 \equiv ax_2 y_1$. y_1 being prime to p, $ax_2 \equiv by_1$.

The lemma is completely proved.

Proof of the Theorem. Taking $r = [p^\epsilon] + 1$ and $s = [p^{1-\epsilon}] + 1$ in Lemma 1, we see that there exists a pair x, y in S satisfying

$$ax \equiv by \pmod{p},$$

where a and b are integers not divisible by the prime p. $[x]$ denotes integral part of x.

Let M be the number of pairs x, y belonging to S. Then

$$M = 2\{[p^\epsilon], [p^{1-\epsilon}]\}.$$

Let S' be the set of pairs x, y such that

$$1 \leqslant x \leqslant p^\epsilon,\ 1 \leqslant y \leqslant p^{1-\epsilon},\ (x, y) = 1.$$

Let N be the numbers of pairs x, y belonging to S' such that corresponding to each pair there corresponds another pair belonging to S which give rise to the same value of b modulo p. Let this set be denoted by S''. So there are exactly $L = (M - N)$ pairs x, y no two pairs of which give rise to the same value of b (mod p). This is to be less than or equal to $p-1$, since there is a one-to-one correspondence between the L pairs and the different values of b which are at the most $p - 1$ in number. But if the lemma is to be true, L must be exactly equal to $p - 1$. For, if $L < p - 1$, then there is at least one b among $1, 2, \ldots, p - 1$ corresponding to which there is no pair in S and thus the lemma fails for this particular b.

So
$$L = M - N = p - 1.$$
That is,
$$\begin{aligned} N &= M - (p-1) \\ &= 2\{[p^{\epsilon}], [p^{1-\epsilon}]\} - (p-1) \\ &= \left(\frac{12}{\pi^2} - 1\right)p + O(p^{\max(1-\epsilon,\epsilon)} \log p) \end{aligned}$$
by Chowla's result.

Since each pair x_1, y_1 belonging to S'' satisfies the equation
$$x_1 |y_2| + x_2 |y_1| = p,$$
where x_2, $|y_2|$ is another pair belonging to S'' and only for pairs belonging to S'' such an equation is satisfied by virtue of Lemma 2, it immediately follows that the number of ways in which p can be represented in the required form is N. The theorem follows.

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Ramanujan Institute
Madras 5

SOME SELF RECIPROCAL FUNCTIONS

By V. V. L. NARASIMHA RAO

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1. The object of this note is to find out a few self reciprocal functions for Hankel transforms. We denote a function $f(x)$ as R_μ, if it is self reciprocal for Hankel transforms of order μ so that, it is given by

$$f(x) = \int_0^\infty J_\mu(xy)\, f(y)\, \sqrt{(xy)}, \tag{1.1}$$

where $J_\mu(x)$ is a Bessel function of order μ. If $\mu = 1/2$, then $f(x)$ is denoted by R_s, while for $\mu = -1/2$, it is denoted by R_c. Hardy and Titchmarsh [13] have shown that under certain broad conditions $f(x)$ is R_μ, if

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{s/2}\, \Gamma\left(\tfrac{1}{4} + \frac{\mu}{2} + \frac{s}{2}\right) \psi(s)\, x^{-s}\, ds, \tag{1.2}$$

where $0 < c < 1$, and $\psi(s) = \psi(1-s)$.

2. We start with the integral given by Bailey [9]

$$\int_0^\infty x^{(\lambda-1)} J_\mu(a\,x)\, J_\nu(b\,x)\, K_\rho(c\,x)\, dx$$

$$= \frac{2^{\lambda-2}\, a^\mu\, b^\nu\, \Gamma\{\frac{1}{2}(\lambda+\mu+\nu-\rho)\}\, \Gamma\{\frac{1}{2}(\lambda+\mu+\nu+\rho)\}}{c^{\lambda+\mu+\nu}\, \Gamma(1+\mu)\, \Gamma(1+\nu)} \times$$

$$\times\, F_4\left(\begin{matrix}\frac{1}{2}(\lambda+\mu+\nu+\rho), & \frac{1}{2}(\lambda+\mu+\nu-\rho) \\ 1+\mu, & 1+\nu;\end{matrix}\; ; -a^2/c^2, -b^2/c^2\right), \tag{2.1}$$

where $R(\lambda+\mu+\nu) > R(\rho)$ and F_4 is Appell's function and each of the four numbers $R(c \pm ai \pm bi)$ is positive. Hence,

$$\int_0^\infty \frac{J_\mu(ax)\, J_\nu(bx)}{x^{\mu+\nu+\frac{1}{2}}}\, K_\rho(cx)\, (cx)^{\lambda+\mu+\nu-\frac{1}{2}}\, dx$$

$$= \frac{2^{\lambda-2}(\sqrt{c})\, a^{\mu} b^{\nu}\, \Gamma\{\tfrac{1}{2}(\lambda+\mu+\nu+\rho)\}\, \Gamma\{\tfrac{1}{2}(\lambda+\mu+\nu-\rho)\}}{(\sqrt{c})\, \Gamma(1+\mu)\, \Gamma(1+\nu)} \times$$

$$\times F_4\left(\begin{matrix}\tfrac{1}{2}(\lambda+\mu+\nu+\rho), & \tfrac{1}{2}(\lambda+\mu+\nu-\rho) \\ 1+\mu, & 1+\nu;\end{matrix}; -a^2/c^2, -b^2/c^2\right), \quad (2.2)$$

where $R(\lambda+\mu+\nu) > R(\rho)$ and each of the four numbers given above is positive.

Finally, putting $\rho = 0$ in (2.2) we obtain that

$$\int_0^\infty \frac{J_\mu(a\,x)\, J_\nu(b\,x)}{x^{\mu+\nu+\frac{1}{2}}} K_0(c\,x)\,(c\,x)^{\lambda+\mu+\nu-\frac{1}{2}}\, dx$$

$$= \frac{2^{\lambda-2}(\sqrt{c})\, a^{\mu} b^{\nu} [\Gamma\{\tfrac{1}{2}(\lambda+\mu+\nu)\}]^2}{(\sqrt{c})\, \Gamma(1+\mu)\, \Gamma(1+\nu)} \times$$

$$\times F_4\left(\begin{matrix}\tfrac{1}{2}(\lambda+\mu+\nu), & \tfrac{1}{2}(\lambda+\mu+\nu) \\ 1+\mu, & 1+\nu;\end{matrix}; -a^2/c^2, -b^2/c^2\right). \quad (2.3)$$

Again, Sneddon [18] has shown that

$$\int_0^\infty J_\mu(ax)\, x^\mu\, e^{-px}\, dx = \frac{\Gamma(\mu+\nu+1)\, a^\nu}{2^\nu \Gamma(1+\nu)\,(a^2+p^2)^{\frac{1}{2}(\mu+\nu+1)}} \times$$

$$\times {}_2F_1\left(\begin{matrix}\tfrac{1}{2}(\mu+\nu+1), \tfrac{1}{2}(\nu-\mu); a^2/(a^2+p^2) \\ \nu+1;\end{matrix}\right), \quad (2.4)$$

where $p > 0$, and $\mu+\nu > 0$.

Hence,

$$\int_0^\infty \frac{J_\nu(ax)}{x^{\nu+\frac{1}{2}}} (px)^{\mu+\nu+\frac{1}{2}} e^{-px}\, dx = \frac{\Gamma(\mu+\nu+1)\, a^\nu p^{\nu+\mu+\frac{1}{2}}}{\Gamma(1+\nu)\,(a^2+p^2)^{\frac{1}{2}(\mu+\nu+1)}} \times$$

$$\times {}_2F_1\left(\begin{matrix}\tfrac{1}{2}(\mu+\nu+1), \tfrac{1}{2}(\nu+\mu); a^2/(a^2+p^2) \\ \nu+1;\end{matrix}\right). \quad (2.5)$$

I have shown in a previous paper [16] that

$$\frac{J_\mu(a_1 x)\, J_\nu(a_2 x) \ldots J_k(a_n x)}{x^{\mu+\nu+\ldots+k+\frac{1}{2}}}, \quad (2.6)$$

where $a_1 + a_2 + \ldots + a_n$ is finite, and $a_1, a_2, \ldots, a_n$ are all positive, is R_1.

Now it is known that (cf. Brij Mohan [1]) the kernel $x^{\nu+\frac{1}{2}} K_0(x)$ transforms R_1 into itself and $x^{3/2} e^{-x}$ transforms R_1 into R_2. Hence

$$f_1(x) = \int_0^\infty \phi(x)\,(xy)^{3/2} K_0(xy)\,dy,$$

$$= \int_0^\infty \frac{J_\mu(a_1 x)\, J_\nu(a_2 x) \ldots J_k(a_n x)}{x^{\mu+\nu+\ldots+k+\frac{1}{2}}} (xy)^{3/2} K_0(xy)\,dy \tag{2.7}$$

is R_1, and also

$$f_2(x) = \int_0^\infty \phi(x)\,(xy)^{3/2} e^{-xy}\,dx$$

$$= \int_0^\infty \frac{J_\mu(a_1 x)\, J_\nu(a_2 x) \ldots J_k(a_n x)}{x^{\mu+\nu+\ldots+k+\frac{1}{2}}} (xy)^{3/2} e^{-xy}\,dy \tag{2.8}$$

is R_2, provided the integrals converge.

The integrals have been evaluated by term by term integration, a process justified under the stated conditions by absolute convergence.

For $n = 2$, we find that (2.7) reduces to (2.3) so that from (2.3) it is clear that the function

$$\frac{1}{\sqrt{(x)}} F_4\left(1, 1; 1+\mu, 1+\nu; -\frac{a^2}{x^2}, -\frac{b^2}{x^2}\right), \tag{2.9}$$

where $\mu+\nu \leqslant 2$, and $R(x \pm a_i + b_i)$ is positive, a and b being positive and finite, is R_1.

For $n = 1$, we find that (2.8) reduces to (2.5), so that from (2.5) it is clear that the function

$$\left(\frac{x}{a^2+x^2}\right)^{3/2} {}_2F_1\left(\begin{matrix}1, \frac{1}{2}\\ 2-\mu\end{matrix}; \frac{a^2}{a^2+x^2}\right), \tag{2.10}$$

where $x > 0$, and a is positive and finite, is R_2.

Finally putting $a = 1$ and $\mu = 1$ in (2.10) we find that

$$\frac{x^{5/2}}{(1+x^2)^2}, \tag{2.11}$$

where $x > 0$, is R_2. Further, it is interesting to note that $f_1(x)$ and $f_2(x)$ generalize the results given by the author in (2.9) and (2.10).

3. Brij Mohan [2] has shown that

$$\int_0^\infty x^{(l-1)} e^{-x^2/2} W_{k,m}(x^2) H_\nu(bx)\, dx$$

$$= \frac{b^{\nu+1}\Gamma(\frac{1}{2}(l+\nu)+m+1)\Gamma(\frac{1}{2}(l+\nu)-m+1)}{2^{\nu+1}(\sqrt{\pi})\Gamma(\nu+3/2)\Gamma(\frac{1}{2}(l+\nu)-k+3/2)} \times$$

$$\times {}_3F_3\left(\begin{matrix}1, \frac{1}{2}(l+\nu)+m+1, \frac{1}{2}(l+\nu)-m+1; -b^2/4\\ 3/2, \nu+3/2, \frac{1}{2}(l+\nu)-k+3/2;\end{matrix}\right), \quad (3.1)$$

where $R(\nu) > -1$, $R(l+\nu \pm 2m) > -2$ and $H_\nu(x)$ is Struve's function of order ν.

Putting $m = 1/4$, $k = -3/8$, and writing $(\nu/2) - \frac{1}{2}$ for ν, and finally using the formula

$$W_{k,\pm\frac{1}{4}}(x) = 2^{\frac{1}{4}-k} x^{1/4} D_{2k-\frac{1}{2}}(2x)^{1/2} \quad (3.2)$$

we find that (3.1) can be written as

$$\int_0^\infty (bx)^l \frac{1}{\sqrt{x}} e^{-x^2/2} D_{-1}(\sqrt{(2)}\, x) H_{(\nu/2-\frac{1}{2})}(bx)\, dx$$

$$= \frac{b^{\nu/2+l+\frac{1}{2}}\Gamma(l/2+\nu/4+1)\Gamma(l/2+\nu/4+\frac{1}{2})}{2^{\nu/2+9/8}(\sqrt{\pi})\Gamma(\nu+1)\Gamma\frac{1}{2}(l/2+\nu/4+13/8)} \times$$

$$\times {}_3F_3\left(\begin{matrix}(1, l/2+\nu/4+1, l/2+\nu/4+\frac{1}{2}; -b^2/4\\ 3/2, \nu/2+1, l/2+\nu/4+13/8;\end{matrix}\right). \quad (3.3)$$

Finally, putting $x/\sqrt{2}$ for x, $\sqrt{(2)}b$ for b, we find that

$$\int_0^\infty (bx)^l \frac{1}{\sqrt{x}} e^{-x^2/4} D_{-1}(x) H_{(\nu/2-\frac{1}{2})}(bx)\, dx$$

$$= \frac{b^{\nu/2+l+\frac{1}{2}}\Gamma(l/2+\nu/4+1)\Gamma(l/2+\nu/4+\frac{1}{2})}{2^{\nu/4-l/2+\frac{1}{8}}(\sqrt{\pi})\Gamma(\nu/2+1)\Gamma(l/2+\nu/2+13/8)} \times$$

$$\times {}_3F_3\left(\begin{matrix}1, l/2+\nu/4+1, l/2+\nu/4+\frac{1}{2}; -b^2/2\\ 3/2, \nu/2+1, l/2+\nu/4+13/8;\end{matrix}\right) \quad (3.4)$$

S. C. Mitra [8] has shown that the function

$$x^{-\mu+\frac{1}{2}}\, e^{-x^2/4}\, D_{-2\mu+1}(x), \tag{3.5}$$

is R_μ. In particular, putting $\mu = 1$ we find that

$$\frac{e^{-x^2/4}\, D_{-1}(x)}{\sqrt{x}}, \tag{3.6}$$

is R_1.

Further, Brij Mohan [1] has shown that the kernel

$$x^{-p/2}\, H_{(p/2-\frac{1}{2})}(x) \tag{3.7}$$

transforms R_1 into R_p.

Applying the results of (3.6), (3.7) and (3.8) to (3.4) we find that

$$(\sqrt{x})\; {}_3F_3\left(\begin{matrix} 1, 1, \frac{1}{2}; -x^2/2 \\ 3/2,\ \nu/2+1,\ 13/8; \end{matrix}\right) \tag{3.8}$$

is R_ν.

4. S. C. Dhar [11] has shown that

$$\int_0^\infty x^{(l-1)}\, x^{x(\frac{1}{2}-\alpha)}\, M_{k_1 m}(x)\, dx$$

$$= \Gamma(l+m+\tfrac{1}{2})\, \alpha^{-l+m+\frac{1}{2}} \times {}_2F_1\left(\begin{matrix} l+m+\frac{1}{2}, -k+m+\frac{1}{2}; \\ 2m+1;\ 1/\alpha. \end{matrix}\right). \tag{4.1}$$

Hence, putting $\alpha = 1$, we find that

$$\int_0^\infty x^{(l-1)}\, e^{-x/2}\, M_{k_1 m}(x)\, dx = \frac{\Gamma(l+m+\frac{1}{2})\,\Gamma(2m+1)\,\Gamma(k-l)}{\Gamma(m+\frac{1}{2}-l)\,\Gamma(m+k+\frac{1}{2})}. \tag{4.2}$$

Applying Mellin's inversion formula [12] we obtain that

$$e^{-x/2}\, M_{k_1 m}(x) = \frac{1}{2\pi i}\int_{c-i}^{c+i\infty} \frac{\Gamma(l+m+\frac{1}{2})\,\Gamma(2m+1)\,\Gamma(k-l)\, x^{-l}\, dl}{\Gamma(m+\frac{1}{2}-l)\,\Gamma(m+k+\frac{1}{2})}, \tag{4.3}$$

where $0 < c < 1$.

Putting $\nu/2+\frac{1}{4}$ for m, $\nu/2+5/4$ for k, and writing $x^2/2$ for x, $l/2$ for l, and taking $0 < c < 1/2 < 1$, we find that (4.3) may be written as

$$e^{-x^2/2} M_{\nu/2+5/4,\nu/2+\frac{1}{4}}(x^2/2)$$

$$= \frac{1}{2\pi i} \int_{2c-i\infty}^{2c+i\infty} 2^{l/2}\, \Gamma(l/2+\nu/2+\tfrac{3}{4})\, \Gamma(\nu+3/2) \times$$

$$\times \frac{\Gamma(\nu/2+5/4-l/2)\, \Gamma(\frac{1}{4}+\nu/2+3/2)\, x^{-l}\, dl}{\Gamma(\nu/2+3/2-l/2)\, \Gamma(m+k+\frac{1}{2})\, \Gamma(\frac{1}{4}+\nu/2+l/2)},$$

$$= \frac{1}{2\pi i} \int_{p-i\infty}^{p+i\infty} 2^{l/2}\, \Gamma(\tfrac{1}{4}+\nu/2+l/2)\, \psi(l)\, x^{-l}\, dl, \tag{4.4}$$

where

$$\psi(l) = \frac{\Gamma(l/2+\nu/2+\frac{3}{4})\, \Gamma(5/4+\nu/2-l/2)\, \Gamma(\nu+3/2)}{\Gamma(\nu/2+3/4-l/2)\, \Gamma(\frac{1}{4}+\nu/2+l/2)} = \psi(1-l).$$

Therefore, from (4.4) and (1.2) it follows that

$$e^{-x^2/4} M_{\nu/2+5/4,\nu/2+\frac{1}{4}}(x^2/2) \tag{4.5}$$

is R_2.

5. Brij Mohan [2] has shown that

$$\int_0^\infty x^{(p-1)}\, e^{-x^2}\, T_m^n(x^2)\, H_\nu(bx)\, dx$$

$$= \frac{b^{\nu+1}\, \Gamma(p/2+\nu/2+\frac{1}{2})\, \Gamma(p/2+\nu/2-m+\frac{1}{2})}{(\sqrt{\pi})\, 2^{\nu+1}\, n\,!\, \Gamma(m+n+1)\, \Gamma(\nu+3/4)\, \Gamma\{\frac{1}{2}(p+\nu+1)-m-n\}}, \tag{5.1}$$

where $R(\nu) > -1$, $R(p+\nu) > -1$, and $R(p+\nu-2m) > -1$.

Putting $x/\sqrt{2}$ for x, $\sqrt{(2)}b$ for b, and $m = n+1$, we find that (5.1) may be put in the form

$$\int_0^\infty (bx)^{p-2n-5/2}\, x^{2n+3/2}\, e^{-x^2/2}\, T_{n+1}^n(x^2/2)\, H_\nu(bx)\, dx$$

$$= \frac{b^{\nu+p-2n-3/2}\, \Gamma\{\frac{1}{2}(p+\nu+1)\}\, \Gamma\{\frac{1}{2}(p+\nu-1)-n\}}{(\sqrt{\pi})\, 2^{\nu/2-p/2-\frac{1}{2}}\, n\,!\, \Gamma(2n+2)\, \Gamma(\nu+3/2)\, \Gamma\{\frac{1}{2}(p+\nu-1)-2n\}}$$

$$\times {}_3F_3\left(\begin{matrix} 1,\ \frac{1}{2}(p+\nu+1),\ \frac{1}{2}(p+\nu-1)-n;\ -b^2/2 \\ 3/2,\ \nu+3/2,\ \frac{1}{2}(p+\nu-1)-2n; \end{matrix}\right) \tag{5.2}$$

where $R(\nu) > -1$, $R(p+\nu) > -1$ and $R(p+\nu-2n) > 1$.

Brij Mohan [3] has shown that the function

$$x^{-2n+3/2}\, e^{-x^2/2}\, T^n_{n+1}(x^2/2), \tag{5.3}$$

where $T^n_{n+1}(x)$ is Sonnine polynomial, is R_1 for all integral values of n.

Applying the results of (5.3) and (3.7) to (5.2) we conclude that

$$x^{\frac{1}{2}}\,{}_3F_3\left(\begin{matrix} 1, p/2+\nu/4+\frac{1}{4}, \frac{1}{2}; -x^2/2 \\ 3/2, \nu/4+5/4, 5/4-\nu/4-p/2; \end{matrix}\right), \tag{5.4}$$

where $\nu + 2p$ is an integer $\geqslant -5$, is R_ν.

6. Brij Mohan [4] has shown that

$$\int_0^\infty x^{(p-1)}\, e^{-ax^2}\, I_\nu(bx)\, dx = \frac{b^\nu\, \Gamma\, \frac{1}{2}(\nu+p)}{2^{\nu+1}\, a^{\nu+p/2}\, \Gamma(\nu+1)} \times$$

$$\times\, {}_1F_1\left(\frac{\nu+p}{2}\,;\, \nu+1\,;\, \frac{b^2}{4a}\right), \tag{6.1}$$

where $R\,(\nu+p) > 0$, and $R(a \pm b) > 0$.

Putting $a = 1/2$, we find that (6.1) may be put in the form

$$\int_0^\infty (bx)^{p-1}\, e^{-x^2/2}\, I_\nu(bx)\, dx = \frac{b^{\nu+p-1}\, \Gamma\{(\nu+p)/2\}}{2^{\nu+1}\, a^{\nu+p/2}\, \Gamma(\nu+1)} \times$$

$$\times\, {}_1F_1\left(\frac{\nu+p}{2}\,;\, \nu+1\,;\, \frac{b^2}{2}\right). \tag{6.2}$$

Brij Mohan [5] has shown that the function

$$e^{-x^2/2}, \tag{6.3}$$

is R_c, and also that the kernel

$$x^{\frac{1}{2}(\lambda+\mu+1)}\, I_{\frac{1}{2}(\mu-\lambda)}(x), \tag{6.4}$$

transforms R_μ into R_λ.

Hence, putting $\mu = -1/2$ we find that the kernel

$$x^{\lambda/2+\frac{1}{4}}\, I_{\frac{1}{2}(-\frac{1}{2}-\lambda/2)}(x) \tag{6.5}$$

transforms R_c to R_λ.

Applying the results of (6.3), (6.5) to (6.2) we find that

$$ {}_1F_1\left(\tfrac{1}{2}; \tfrac{3}{4} - \frac{\lambda}{2}; \frac{x^2}{2}\right), \tag{6.6} $$

is R_λ. It is interesting to compare this function with the R_μ function

$$ {}_1F_1\left(\tfrac{1}{2}; \frac{5}{4} - \frac{\nu}{2}; \frac{x^2}{2}\right), \tag{6.7} $$

given by Brij Mohan [7].

7. In a previous paper [17] I have shown that the function

$$ x^{\alpha_1 - \frac{1}{2}} {}_{q-1}F_q\Big(\alpha_1, \tfrac{1}{2}(\alpha_1 + m_1 + 1), \tfrac{1}{2}(\alpha_1 + m_2 + 1), \ldots \tfrac{1}{2}(\alpha_1 + m_{q-2} + 1); -x^2/4 \\ \tfrac{1}{2}(\alpha_1 - m_1 + 1), \tfrac{1}{2}(\alpha_1 - m_2 + 1), \ldots \tfrac{1}{2}(\alpha_1 - m_{q-2} + 1), \\ \tfrac{1}{2}(\alpha_1 - \lambda + 1), \tfrac{1}{2}(\alpha_1 + \nu + 1)\Big) \tag{7.1} $$

transforms R_λ into $\pm R_\nu$ according as $m_1 + m_2 + \ldots + m_{q-2} + \lambda$ is an even or odd integer and $m_1, m_2, \ldots, m_{q-2}$ and λ are all positive integers.

Hence, using the formula

$$ E(p; \alpha_s: q; \rho_s: x) = \frac{\Gamma(\alpha_1)\,\Gamma(\alpha_2) \ldots, \Gamma(\alpha_\beta)}{\Gamma(\rho_1)\,\Gamma(\rho_2) \ldots \Gamma(\rho_q),}\, {}_1F_1\left(p; \alpha_r: q, \rho_s; -\frac{1}{x}\right) \tag{7.2} $$

we find that the kernel

$$ x^{\alpha_1 - \frac{1}{2}} E\Big(\alpha_1, \tfrac{1}{2}(\alpha_1 + m_1 + 1), \tfrac{1}{2}(\alpha_1 + m_2 + 1), \ldots \tfrac{1}{2}(\alpha_1 + m_{q-2} + 1): 4/x^2 \\ \tfrac{1}{2}(\alpha_1 - m_1 + 1), \tfrac{1}{2}(\alpha_1 - m_2 + 1), \ldots \tfrac{1}{2}(\alpha - m_{q-2} + 1), \\ \tfrac{1}{2}(\alpha_1 - \lambda + 1), \tfrac{1}{2}(\alpha_1 + \nu + 1)\Big) \tag{7.3} $$

transforms R_λ into $\pm R_\nu$ according as $m_1 + m_2 + \ldots + m_{q-2} + \lambda$ is an even or odd integer, where $m_1, m_2 \ldots, m_{q-2}$ and λ are all positive integers. Macrobert [15] has shown that

$$ E(\alpha_1, \alpha_2, \ldots, \alpha_r, r, 2\nu + r : \beta_1\, \beta_2 \ldots \beta_s, \nu + r - l + \tfrac{1}{2} : p) \\ = \int_0^\infty e^{-\mu/2}\, \mu^{\nu+\gamma-3/2}\, W_{l,\nu}(\mu)\, E\left(\alpha_1, \alpha_2, \ldots, \alpha_r : \beta_1\, \beta_2 \ldots \beta_s : \frac{p}{\mu}\right) d\mu, \tag{7·4} $$

where $R(\gamma) > 0$, $R(2\nu+\gamma) > 0$, and $R(p) > 0$.

Putting $\mu^2/2$ for μ, $2/p^2$ for p and $r = (s-1)$, we find that

$$E(\alpha_1\,\alpha_2, \ldots \alpha_{s-1}, \gamma, 2\nu+\gamma : \beta_1, \beta_2, \ldots, \beta_s, \nu+r-l+\tfrac{1}{2} : 2/p^2)$$
$$= \int_0^\infty e^{-\mu^2/4}\, \frac{\mu^{2\nu+2\gamma-2}}{2^{\nu+\gamma-3/2}}\, W_{l,\nu}(\mu^2/2) E(\alpha_1,\alpha_2,\ldots,\alpha_{s-1} : \beta_1, \beta_2,\ldots,\beta_s : 4/p^2\mu^2)du.$$

Further, on putting $\nu = \frac{1}{4}$ and writing $l-\frac{1}{4}$ for l and using the formula

$$W_{l,\frac{1}{4}}(x) = 2^{-l+\frac{1}{4}}\, x^{\frac{1}{4}}\, D_{2l-\frac{1}{2}}(\sqrt{(2x)}), \tag{7.5}$$

we obtain that

$$\int_0^\infty e^{-\mu^2/4}\, D_{2l-1}(\mu)\, \mu^{2r-1}\, E(\alpha_1, \alpha_2, \ldots, \alpha_{s-1} : \beta_1, \beta_2 \ldots, \beta_s : 4/p^2\mu^2)du$$
$$= 2^{l+\gamma-1}\, E(\alpha_1, \alpha_2, \ldots, \alpha_{s-1}, \gamma, \gamma+\tfrac{1}{2} : \beta_1\,\beta_2 \ldots \beta_s, \gamma-l+\tfrac{1}{4} : 2/p^2),$$

where $R(r) > 0$, and $R(p) > 0$; so that

$$\int_0^\infty e^{-\mu^2/4}\, D_{2l-1}(u)\, \mu^{2\gamma-\alpha_1-\frac{1}{2}}\, (up)^{\alpha_1-\frac{1}{2}} \times$$
$$\times E(\alpha_1, \alpha_2, \ldots, \alpha_{s-1} : \beta_1, \beta_2, \ldots \beta_s : 4/p^2\mu^2)\, du$$
$$= 2^{l+\gamma+1}\, p^{\alpha_1-\frac{1}{2}}\, E\begin{pmatrix} \alpha_1, \alpha_2, \ldots, \alpha_{s-1}, \gamma, \gamma+\frac{1}{2} : 2/p^2 \\ \beta_1\,\beta_2 \ldots \beta_s, \gamma-l+1 : \end{pmatrix}. \tag{7.6}$$

Now let $\alpha_2 = \frac{1}{2}(\alpha_1+m_1+1)$, $\alpha_3 = \frac{1}{2}(\alpha_1+m_2+1), \ldots,$

$$\alpha_{s-1} = \tfrac{1}{2}(\alpha_1+m_{s-2}+1)$$

and

$$\beta_1 = \tfrac{1}{2}(\alpha_1-m_1+1),\ \beta_2 = \tfrac{1}{2}(\alpha_1-m_2+1), \ldots,$$
$$\beta_{s-2} = \{\tfrac{1}{2}(\alpha_1-m_{s-2}+1)\}$$
$$\beta_{s-1} = \tfrac{1}{2}(\alpha_1-l+1) \text{ and } \beta_s = \tfrac{1}{2}(\alpha_1+\nu+1),$$

where $m_1, m_2, \ldots, m_{s-2}$ and l are all positive integers.

Also let $2\gamma-\alpha_1 = l$. Then we find that (7.6) may be written as

$$\int_0^\infty e^{-\mu^2/4}\, D_{2l-1}(\mu)\, \mu^{2\gamma-\alpha_1-\frac{1}{2}}(\mu p)^{\alpha_1-\frac{1}{2}} \times$$
$$\times E\begin{pmatrix} \alpha_1, (\alpha_1+m_1+1)/2, \ldots (\alpha_1+m_{s-2}+1)/2 : (\alpha_1-m_1+1)/2, \ldots \\ (\alpha_1-m_{s-2}+1)/2, (\alpha_1-l+1)/2, (\alpha_1+\nu+1)/2 : 4/p^2\mu^2 \end{pmatrix} d\mu$$

$$= 2^{3\gamma-\alpha_1-1}p^{\alpha_1-\frac{1}{2}}E\Big(\alpha_1,(\alpha_1+m_1+1)/2,\ldots(\alpha_1+m_{s-2}+1)/2,$$
$$(\alpha_1+l)/2,(\alpha_1+l+1)/2:(\alpha_1-m_1+1)/2,\ldots\alpha$$
$$(\alpha_1-m_{s-2}+1)/2,(\alpha_1-l+1)/2,(\alpha_1+\nu+1)/2:2/p^2\Big). \tag{7.7}$$

S. C. Mitra [14] has shown that

$$e^{-x^2/4}\,x^{l-\frac{1}{2}}\,D_{2l-1}(x) \tag{7.8}$$

is $\pm\, R_l$ according as $(l-1)$ is even or odd.

Hence, from (7.3), (7.7) and (7.8) it follows that

$$x^{\alpha_1-\frac{1}{2}}E\Big(\alpha_1,(\alpha_1+m_1+1)/2,\ldots(\alpha_1+m_{s-2}+1)/2,(\alpha_1+l+1)/2,(\alpha_1+l)/2:$$
$$(\alpha_1-m_1+1)/2,(\alpha_1-m_2+1)/2,\ldots(\alpha_1-m_{s-2}+1)/2,$$
$$(\alpha_1-l+1)/2,(\alpha_1+\nu+1)/2,(\alpha_1-l)/2+1:2/x^2\Big) \tag{7.9}$$

is $\pm\, R_\nu$ according $m_1+m_2+\ldots+m_{s-2}+l$ is even or odd.

In case $l=1$, we find that

$$x^{\alpha_1-\frac{1}{2}}E\left(\begin{matrix}\alpha_1,(\alpha_1+m_1+1)/2,\ldots(\alpha_1+m_{s-2}+1)/2,(\alpha_1+2)/2:-x^2/2\\(\alpha_1-m_1+1)/2,\ldots(\alpha_1+m_{s-2}+1)/2,\alpha_1/2,(\alpha_1+\nu+1)/2:\end{matrix}\right)$$

is $\pm\, R_\nu$ according as $m_1+m_2+m_{s-2}+1$ is even or odd. But this is the same R_ν function

$$x^{\alpha_1-\frac{1}{2}}\,{}_sF_s\Big(\alpha_1,(\alpha_1+m_1+1)/2,\ldots(\alpha_1+m_{s-2}+1)/2,(\alpha_1+2)/2;-x^2/2$$
$$(\alpha_1-m_1+1)/2,(\alpha_1-m_2+1)/2,\ldots,$$
$$(\alpha_1-m_{s-2}+1)/2,(\alpha_1/2,\alpha_1+\nu+1)/2)\Big), \tag{7.10}$$

given by Dineschandra [10] which is $\pm\, R_\nu$ according as $m_1+m_2+\ldots+m_{s-2}+1$ is even or odd.

8. Watson [19] has shown that

$$\int_0^\infty J_\mu(at)\,J_\nu(at)\,e^{-p^2t^2}t^{\lambda-1}\,dt = \frac{a^{\mu+\nu}\,\Gamma\{\frac{1}{2}(\lambda+\mu+\nu)\}}{2^{\mu+\nu}\,p^{\lambda+\mu+\nu}\,\Gamma(1+\mu)\,\Gamma(1+\nu)}\times$$
$$\times\,{}_3F_3\left(\begin{matrix}(\mu+\nu+1)/2,(\mu+\nu+2)/2,(\lambda+\mu+\nu)/2;-a^2/2\\\mu+1,\ \nu+1,\ \mu+\nu+1;\end{matrix}\right). \tag{8.1}$$

Putting $\nu - \frac{1}{2}$ for ν and μ, $p = \sqrt{2}$ and writing $t/2$ for t, we find that (8.1) may as well be written as

$$\int_0^\infty J_{\nu-\frac{1}{2}}(at/2)\, J_{\nu-\frac{1}{2}}(at/2)\, e^{-t^2/2}\, t^{\lambda-\mu-3/2} (at)^{\nu+\frac{1}{2}}\, dt$$

$$= \frac{a^{3\nu-\frac{1}{2}}\, \Gamma(\lambda + 2\nu - 1)}{2^{3\nu-\lambda/2-3/2}[\Gamma(\nu+\frac{1}{2})]^2} \times {}_2F_2\left(\begin{matrix} \nu, \frac{1}{2}(\lambda + 2\nu - 1); -a^2/2 \\ \nu + \frac{1}{2}, 2\nu; \end{matrix} \right). \tag{8.2}$$

Hence, on taking $\lambda = 4\nu + 1$ we find that it may be written as

$$\int_0^\infty J_{\nu-\frac{1}{2}}(at/2)\, J_{\nu-\frac{1}{2}}(at/2)\, e^{-t^2/2}\, t^{3\nu-\frac{1}{2}} (at)^{\nu+\frac{1}{2}}\, dt$$

$$= \frac{a^{3\nu-\frac{1}{2}}\, \Gamma(6\nu)}{2^{\nu-2}[\Gamma(\nu + \frac{1}{2})]^2} \times {}_2F_2\left(\begin{matrix} \nu, 3\nu; -a^2/2 \\ \nu + \frac{1}{2}, 2\nu; \end{matrix} \right). \tag{8.3}$$

Brij Mohan [1] has shown that the kernel

$$J_{\nu-\frac{1}{2}}(t/2)\, J_{\nu-\frac{1}{2}}(t/2)\, t^{\nu+\frac{1}{2}} \tag{8.4}$$

transforms $R_{3\nu-\frac{1}{2}}$ into R_ν.

Hence, applying the results of (7.8), (8.3) we conclude that

$$x^{3\nu-\frac{1}{2}}\, {}_2F_2\left(\begin{matrix} \nu, 3\nu; -x^2/2 \\ \nu + \frac{1}{2}, 2\nu; \end{matrix} \right), \tag{8.5}$$

is R_ν.

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Mushirabad
Hyderabad (Deccan)-20

ON FOUR INTERSECTING SPHERES

By SAHIB RAM MANDAN

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ABSTRACT*

The interesting results 'On Three Intersecting Circles' obtained by Prof. N.A. Court[4] led me to the present paper. The following results of some interest have been arrived at:

The radical tetrahedron of four intersecting spheres coincides with a diagonal tetrahedron of the desmic system of intersection of those spheres. The pairs of opposite vertices of the system referred to this tetrahedron form pairs of conjugate points for the orthogonal sphere of the given four spheres and are the centres of similitude of the tetrad of associated spheres.

The planes of perspectivity of the eight pairs of complementary tetrahedra of intersection of four intersecting spheres form two tetrahedra desmic with their radical tetrahedron, and are the radical planes of the corresponding pairs of complementary spheres of intersection, and are the planes of similitude of the associated tetrad of spheres.

The diagonal tetrahedra of a desmic system of intersection of four intersecting spheres form the other desmic system of intersection of those spheres.

The eight centres of similitude (other than the orthogonal centre of four intersecting spheres) of the eight pairs of complementary spheres of intersection of the four given spheres form two tetrahedra desmic with their central tetrahedron and thus form a set of eight associated points.

The desmic system of intersection of four intersecting spheres is inscribed in the one conjugate to that of centres for those spheres and reciprocally their other desmic system of intersection is circumscribed to that of centres for them.

The perpendicular from the orthogonal centre of four intersecting spheres upon the Newtonian plane of their associated tetrad of spheres passes through the circumcentre of their radical tetrahedron.

Each sphere of anti-similitude of the associated tetrad of spheres is orthogonal to eight of the spheres of intersection and to two of the four given intersecting spheres.

In the end corresponding results for four mutually orthogonal and real spheres are deduced.

Finally umbilical projection is suggested as a process to get these results rather quickly.

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1. Introduction. a. Given four mutually intersecting spheres (A), (B), (C), (D), let (M) be their orthogonal sphere, and let

$$P, P'; \ Q, Q'; \ R, R'; \ T, T' \qquad (a)$$

be the pairs of points of intersection of the triads of the spheres $(B), (C), (D)$; $(C), (D), (A)$; $(D), (A), (B)$; $(A), (B), (C)$ respectively.

The points of (a) located on each of the given spheres are listed in the following [5]

TABLE 1.

(A)		Q, Q';	R, R';	T, T'.
(B)	P, P';		R, R';	T, T'.
(C)	P, P';	Q, Q';		T, T'.
(D)	P, P';	Q, Q';	R, R'.	

b. In what follows we shall consider the relations between the tetrahedra and spheres determined by the points of intersection (a) of the given spheres.

In connection with this study it will be purposeful to consider the tetrahedron $(T') = m_a\, m_b\, m_c\, m_d$ formed by the radical planes, m_a, m_b, m_c, m_d, of the pairs of spheres $(M), (A)$; $(M), (B)$; $(M), (C)$; (M), (D) respectively. We shall refer to (T') as the *radical tetrahedron* of the spheres (A), (B), (C), (D).

The tetrahedron $(T) = ABCD$ formed by the centres of the four given spheres shall be said to be the *central tetrahedron* of those spheres.

The vertices $A' = m_b\, m_c\, m_d$, $B' = m_c\, m_d\, m_a$, $C' = m_d\, m_a\, m_b$, $D' = m_a\, m_b\, m_c$ of the radical tetrahedron lie, respectively, on the radical axes $p = PP'$, $q = QQ'$, $r = RR'$, $t = TT'$ of the given spheres taken three at a time. It is readily seen that the radical tetrahedron and the central tetrahedron are polar reciprocals w.r.t. the sphere (M). Therefore AA', BB', CC', DD' generate a quadric [2].

2. Perspective Tetrahedra. a. Taking one point from each of the four pairs of points of intersection (a) we obtain sixteen

tetrahedra which we shall refer to as the *tetrahedra of intersection* [5]. They are listed in the following

TABLE 2.

(1)	(2)	(3)	(4)
$PQRT$,	$PQ'R'T'$,	$P'QR'T'$,	$P'Q'RT'$,
$P'Q'R'T'$;	$P'QRT$;	$PQ'RT$;	$PQR'\ T$;

(5)	(6)	(7)	(8)
$P'Q'R'T$,	$PQR'T'$,	$P'QRT'$,	$PQ'RT'$,
$PQRT'$;	$P'Q'RT$;	$PQ'R'T$;	$P'QR'T$;

The sixteen tetrahedra of intersection are grouped into eight pairs of *complementary* [5] *tetrahedra of intersection.* Two such tetrahedra of a pair have no vertex in common, and thus involve all eight points of intersection.

b. The four radical axes p, q, r, t (Art. 1b) meet in the orthogonal centre M of the spheres (A), (B), (C), (D), hence the eight pairs of complementary tetrahedra of intersection are in perspective. Let s, s_p, s_q, s_r, s_t, s_{pq}, s_{qr}, s_{rp} denote their respective planes of perspectivity.

Let

$$U = (QR,\ Q'R'),\ \ V = (RP,\ R'P'),\ W = (PQ,\ P'Q'),$$
$$X = (PT,\ P'T'),\ Y = (QT,\ Q'T'),\ Z = (RT,\ R'T'),$$
$$U' = (QR',\ Q'R),\ \ V' = (RP',\ R'P),\ W' = (PQ',\ P'Q),$$
$$X' = (PT',\ P'T),\ Y' = (QT',\ Q'T),\ \ Z' = (RT',\ R'T);$$

$$a = (QR'T',\ Q'RT),\ \ b = (Q'RT',\ QR'T),$$
$$c = (Q'R'T,\ QRT'),\ \ d = (QRT,\ Q'R'T'),$$
$$e = (RT'P',\ R'TP),\ \ f = (R'TP',\ RT'P),$$
$$g = (R'T'P,\ RTP'),\ \ h = (RTP,\ R'T'P'),$$
$$e' = (TP'Q',\ T'PQ),\ \ f' = (T'PQ',\ TP'Q)$$
$$g' = (T'P'Q,\ TPQ'),\ \ h' = (TPQ,\ T'P'Q'),$$
$$a' = (PQ'R',\ P'QR),\ \ b' = (P'QR',\ PQ'R),$$
$$c' = (P'Q'R,\ PQR'),\ \ d' = (PQR,\ P'Q'R').$$

Then we have, by Desargue's theorem,

$$a = U'Y'Z, \quad b = U'YZ', \quad c = UY'Z', \quad d = UYZ,$$
$$e = XV'Z', \quad f = X'VZ', \quad g = X'V'Z, \quad h = XVZ,$$
$$e' = X'Y'W, \quad f' = X'YW', \quad g' = XY'W', \quad h' = XYW,$$
$$a' = UV'W', \quad b' = U'VW', \quad c' = U'V'W, \quad d' = UVW\,;$$

and

$$s = dd'hh' = UVWXYZ, \quad s_p = a'df'g = UV'W'X'YZ,$$
$$s_q = ab'g'h = U'VW'XY'Z, \quad s_r = bc'eh' = U'V'WXYZ',$$
$$s_t = cd'e'f = UVWX'Y'Z', \quad s_{pq} = ac'e'g = U'V'WX'Y'Z,$$
$$s_{qr} = a'ceg' = UV'W'XY'Z', \quad s_{rp} = bb'ff' = U'VW'X'YZ'.$$

3. The Radical Tetrahedron. a. The four points Q, Q', R, R' determine a complete quadrangle inscribed in the circle common to the spheres (A) & (D) (see Table 1, Art. 1a) having the points M, U, U' for vertices of its diagonal triangle (Art. 2b). Hence the points U, U' lie on the polar line of M w.r.t. the said circle [4]. Similarly Y, Y' and Z, Z' lie respectively on the polar lines of M w.r.t. the circles of the sphere (A) common to the spheres (C) & (B). Thus the 3 lines UU', YY', ZZ' lie in the polar plane of M w.r.t. the sphere (A). Now the spheres (M), (A) being orthogonal (Art. 1a), the polar plane of M for (A) coincides with the radical plane m_a (Art. 1b) of those two spheres. Hence the plane m_a contains the complete quadrilateral $abcd$ (Art. 2b) with UU', YY', ZZ' as its three diagonals.

Similarly the planes m_b, m_c, m_d, contain respectively the quadrilaterals $efgh$, $e'f'g'h'$, $a'b'c'd'$ with their diagonals as XX', VV', ZZ'; XX', YY', WW'; UU', VV', WW'.

b. Thus XX', UU'; YY', VV'; ZZ', WW' form the three pairs of opposite edges of the radical tetrahedron $(T') = A'B'C'D' = m_a m_b m_c m_d$ (Art. 1b.), and thence $B'C' = m_a m_d = UU'$, $C'A' = m_b m_d = VV'$, $A'B' = m_c m_d = WW'$, $A'D' = m_b m_c = XX'$, $B'D' = m_c m_a = YY'$, $C'D' = m_a m_b = ZZ'$; $A' = (XX', VV', WW')$, $B' = (UU', YY', WW')$, $C' = (UU', VV', ZZ')$, $D' = (XX', YY', ZZ')$.

c. It follows then that $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$ are respectively the diagonal triangles of the quadrilaterals *abcd*, *efgh*, *e'f'g'h'*, *a'b'c'd'* in the faces of the tetrahedron (T'). Hence by the harmonic property of the complete quadrilaterals, we may note that U,U'; V,V'; W,W'; X,X'; Y,Y'; Z,Z' are six pairs of points on the six edges of (T') separating harmonically the respective pairs of its vertices, or in other words the radical tetrahedron is self-polar for these six pairs of points.

d. The two vertices U,U' of the diagonal triangle MUU' of the complete quadrangle $QQ'RR'$ (Art. 3a) are conjugate w.r.t. its circumcircle and therefore for the sphere (A), and lie on the radical plane m_a of the two (orthogonal) spheres (M), (A). Hence U,U' are also conjugate for, or in-polar to, the sphere (M), for they separate harmonically the intersections of their join UU' with the circle, in the plane m_a, common to the spheres (A) and (M).

Similarly are the other five pairs of points V,V'; W,W'; X,X'; Y,Y'; Z,Z' in-polar to the sphere (M).

4. Desmic Systems of Intersection. a. X, Y, Z, U, V, W are noticed to lie in the plane s (Art. 2b) as the vertices of the quadrilateral *dd'hh'* and therefore as the traces of this plane on the edges of the tetrahedron (T'), while X', Y', ..., W' are their harmonic conjugates for the corresponding pairs of its vertices (Art. 3c). Therefore the twelve points X, Y, ..., W' form a desmic system (S) with (T') as its diagonal tetrahedron and X, X'; Y, Y'; ... ; W, W' as the pairs of its opposite vertices [3] referred to this tetrahedron.

We shall refer to this system as the *desmic system of intersection* of the given four spheres.

Hence : *The radical tetrahedron of four intersecting spheres coincides with a diagonal tetrahedron of the desmic system of intersection of those spheres. The pairs of opposite vertices of the system referred to this tetrahedron form pairs of conjugate points w.r.t. the orthogonal sphere of the four given spheres* (Art. 3d).

b. In fact, Arts. 3c and 3d are enough to establish the desmic system(S) [7].

c. The eight planes of perspectivity (Art. 2b) form two tetrahedra $(T'') = ss_{pq}s_{qr}s_{rp}$, $(T''') = s_p s_q s_r s_t$ desmic with the tetrahedron (T') (Art. 3a), for through the line of intersection of any face of one tetrahedron with any face of one of the other two tetrahedra passes a face of the third tetrahedron [2] (Art. 2b).

Hence : *The planes of perspectivity of the eight pairs of complementary tetrahedra of intersection of four intersecting spheres form two tetrahedra desmic with their radical tetrahedron.* We shall refer to these three tetrahedra to form a *second desmic system* (S') *of intersection* of the four given spheres.

d. Let $A'' = ss_{pq}s_{rp}$, $B'' = ss_{pq}s_{qr}$, $C'' = ss_{qr}s_{rp}$, $D'' = s_{pq}s_{qr}s_{rp}$; $A''' = s_q s_r s_t$, $B''' = s_r s_t s_p$, $C''' = s_t s_p s_q$, $D''' = s_p s_q s_r$. Then by Art. 2b we have $A'' = (U'X', VY, WZ)$, $B'' = (UX, V'Y', WZ)$, $C'' = (UX, VY, W'Z')$, $D'' = (U'X', V'Y', W'Z')$; $A''' = (U'X, VY', WZ')$, $B''' = (UX', V'Y, WZ')$, $C''' = (UX', VY', W'Z)$, $D''' = (U'X, V'Y, W'Z)$. We may now notice here that $A''B''C''$ is the diagonal triangle of the quadrilateral $dd'hh'$ (Art. 2b) in the plane s and hence the pairs of points U, X; V, Y; W, Z lie on the sides $B''C''$, $C''A''$, $A''B''$ separated harmonically by the respective pairs of vertices, viz. B'', C''; C'', A''; A'', B''.

Similarly we notice that the pairs of points U', X'; V', Y'; W', Z' lie on the edges $A''D''$, $B''D''$, $C''D''$, of the tetrahedron (T''), separated harmonically by the respective pairs of its vertices, viz. A'', D''; B'', D''; C'', D''.

e. Further we notice that U, Y, W, X', Y', Z' lie in the plane s_t (Art. 2b) as the traces of this plane on the edges of the tetrahedron (T''), and X, Y, Z, U', V', W' are their harmonic conjugates for the corresponding pairs of its vertices (Art. 4d). Therefore by the above argument (Art. 4a) we have (T'') as a diagonal tetrahedron of the desmic system (S).

f. Similarly for the tetrahedron (T'''). Again from symmetry or otherwise we may observe the tetrahedra $UU'XX'$, $VV'YY'$, $WW'ZZ'$ of desmic system (S) to form the three diagonal tetrahedra of the second desmic system (S').

Thus : *The tetrahedra of a desmic system of intersection of four intersecting spheres are the three diagonal tetrahedra of the other desmic system of intersection of the four given spheres.* Or, in other words : *The diagonal tetrahedra of a desmic system of intersection of four intersecting spheres form the other desmic system of intersection of these spheres.*

Hence : *The two desmic systems of intersection of four intersecting spheres are conjugate* [7].

5. Spheres of Intersection. a. The circumspheres of the tetrahedra of intersection (Art. 2a) will be said to be the *spheres of intersection* [5] of the four given spheres.

Two spheres of intersection will be said to be *complementary* [5], if their inscribed tetrahedra of intersection are complementary.

b. The points P, P' lie on the sphere (B) (See Table 1) and are collinear with the centre M of the sphere (M) orthogonal to (B). Hence P, P' are inverse points [2] w.r.t. (M). The pairs of points Q, Q' ; R, R' ; T, T' are inverse w.r.t. (M), for analogous reasons.

Thus the spheres $(I) = PQRT, (I') = P'Q'R'T'$ are inverses w.r.t. (M), and (M) is a sphere of anti-similitude [2] of (I), (I').

Similarly for the other pairs of complementary spheres of intersection of the given spheres. Thus : *The orthogonal sphere of four intersecting spheres is a sphere of anti-similitude of each of the eight pairs of complementary spheres of intersection of the given spheres.*

c. The points P, P' of the spheres $(I), (I')$, inverses w.r.t. (M), are anti-homologous points on these two spheres [4]. Similarly for the pairs of points Q,Q' ; R,R' ; T,T'. Hence the pairs of corresponding faces of two complementary tetrahedra of intersection $PQRT$, $P'Q'R'T'$ are anti-homologous planes of the two spheres (I), (I') and therefore intersect on their radical plane [2]. Thus the radical plane of (I), (I') coincides with the plane of perspectivity s of the tetrahedra $PQRT$, $P'Q'R'T'$.

Similarly for the other pairs of complementary spheres of intersection. Hence : *The planes of perspectivity of the eight pairs of*

complementary tetrahedra of intersection of four intersecting spheres are the radical planes of the corresponding pairs of complementary spheres of intersection [5] *of the given spheres.*

d. The three spheres (M), (I), (I') being coaxal (Art. 5b), their three centres M,I,I' are collinear. Now M is the centre of perspectivity of the two tetrahedra $PQRT$, $P'Q'R'T'$, and their plane of perspectivity coincides with the radical plane of the two spheres (I), (I') (Art. 5c). Hence : *The line joining the circumcentres of the two complementary tetrahedra of intersection of four intersecting spheres passes through the centre of perspectivity of the two tetrahedra and is perpendicular to their plane of perspectivity.*

e. It follows from the preceding proposition (Art. 5d), that u, u_i $(i = p,q,r,t,pq,qr,rp)$ are perpendiculars from M upon the planes of perspectivity s, s_i (Art. 2b) respectively. We have the following triads of collinear points: $u = MII'$, $u_i = MI_i I_i'$, where I_i, I_i' are the centres of the pairs of complementary spheres of intersection, other than I, I', in correspondence with s_i.

6. The Centres of Similitude of the Spheres of Intersection. a. The centre M of the sphere of anti-similitude (M) of the spheres (I), (I') (Art. 5b) is a centre of similitude of those two spheres. Hence their second centre of similitude is the harmonic conjugate M_0 of M w.r.t. their centres I, I', and the sphere (M_0) having M_0 for centre and coaxal with (I), (I') is their second sphere of anti-similitude. Now the two spheres of anti-similitude (M), (M_0) are orthogonal [9]. Hence the centre M_0 of (M_0) coincides with the pole, for (M), of the radical plane s of the two spheres (Art. 5c).

It may be shown in analogous fashion that the second centres of similitude, besides the point M, of the pairs of complementary spheres of intersection (cf. Table 2), other than (I), (I'), are the poles M_i of their respective radical planes s_i (Art. 5e) w.r.t. the sphere (M). Thus : *Given four intersecting spheres, any two of their complementary spheres of intersection have for their two centres of similitude:* (*i*) *the centre of the orthogonal sphere of the four given spheres, and* (*ii*) *the*

pole, for this sphere, of the radical plane of the two spheres of intersection considered.

b. The two tetrahedra $M_0 M_{pq} M_{qr} M_{rp}$, $M_p M_q M_r M_t$ formed by the eight points M, M_i (Art. 6a) are polar reciprocals, for the sphere (M), of the tetrahedra (T''), (T''') which are desmic with the radical tetrahedron (T') (Art. 4c) that again is the polar reciprocal, for (M), of the central tetrahedron (T) (Art. 1b) of the four given spheres. Hence : *The eight centres of similitude (other than the orthogonal centre of four intersecting spheres) of the eight pairs of complementary spheres of intersection of the four given spheres form two tetrahedra desmic with their central tetrahedron and thus form a set of eight associated points*[6].

The desmic system (S'') introduced here is the polar reciprocal of the one (S') introduced already (Art. 4c) for the sphere (M). We shall call (S'') as *desmic system of centres* for the given four spheres.

c. If (S''') is the desmic system conjugate (Art. 4f) to (S''), it is readily seen that (S''') is the polar reciprocal, for the sphere (M), of the desmic system (S) (Art. 4a), i.e. a face of (S''') is the polar plane, for (M), of a vertex of (S) and therefore passes through its conjugate vertex (Art. 3d) for (M). Thus : *The desmic system of intersection for four intersecting spheres is inscribed in the one conjugate to that of centres for these spheres, and reciprocally their other desmic system of intersection is circumscribed to that of centres for them.*

7. The Associated Tetrad of Spheres. a. The point of intersection A' of the three radical planes m_b, m_c, m_d (Art. 1b) is the orthogonal centre of the four spheres (M), (B), (C), (D). Let (A') denote their orthogonal sphere.

We have similarly the orthogonal spheres (B'), (C'), (D') of the three tetrads of spheres (M), (C), (D), (A) ; (M), (D), (A), (B) ; (M), (A), (B), (C).

The spheres (A'), (B'), (C'), (D') form the *associated tetrad of spheres* of the four given spheres.

From the preceding relations and Art. 1 it follows that the spheres (A), (B), (C), (D), respectively are the orthogonal spheres of the tetrads of spheres (M), (B'), (C'), (D') ; (M), (C'), (D'), (A') ; (M), (D'), (A'), (B') ; (M), (A'), (B'), (C').

b. The two points U, U' being conjugate w.r.t. (M) (Art. 3d), the sphere (UU') having UU' for diameter is orthogonal[2] to (M). The centre of (UU') obviously lies on the line $UU' = m_a m_d = B'C'$ (Art. 3b), hence the three spheres (B'), (C'), (UU'), with collinear centres and orthogonal to the same sphere (M), are coaxal[2].

Moreover, U, U' being separated harmonically by B', C' (Art. 3c) (UU') is the sphere of similitude[2] and U, U' are the centres of similitude[2] of the spheres (B'), (C').

Likewise the analogous spheres (VV'), (WW'), (XX'), (YY'), (ZZ') are, respectively, the spheres of similitude of the pairs of spheres (C'), (A'); (A'), (B'); (A'), (D'); (B') (D'); (C'), (D'), Thus: *The pairs of opposite vertices of the desmic system of intersection of four intersecting spheres referred to their radical tetrahedron are the six pairs of centres of similitude of the tetrad of associated spheres, of the given spheres, taken two by two.*

It follows that: *The planes of perspectivity* (Art. 2b) *of the pairs of complementary tetrahedra of intersection of four intersecting spheres are the eight planes of similitude* [2] *of their associated tetrad of Spheres.*

c. The centres of the six spheres of similitude (Art. 7b) of the four spheres (A'), (B'), (C'), (D') taken in pairs lie in their Newtonian plane [2], which is the radical plane of their orthogonal sphere (M) and their central sphere $(o') = A'B'C'D'$. Thus: *The perpendicular from the orthogonal centre of four intersecting spheres upon the Newtonian plane of their associated tetrad of spheres passes through the circumcentre of their radical tetrahedron.*

8. Spheres of Anti-Similitude. a. The spheres (U), (U') coaxal with the spheres (B'), (C') and having for centres their centres of similitude U, U' are their two spheres of anti-similitude.

The analogous pairs of spheres (V), (V'); (W), (W'); (X), (X'); (Y), (Y'); (Z), (Z') are the spheres of anti-similitude of the pairs of spheres (C'), (A'); (A'), (B'); (A'), (D'); (B'), (D'); (C'), (D') respectively.

b. The sphere (M), being orthogonal to (B') and (C') (Art. 7a) is also orthogonal to the spheres (U), (U') coaxal with them. The sphere (U), therefore, having its centre on the radical plane of the three coaxal spheres (I), (I'), (M) (Art. 5c), is orthogonal to both (I) and (I'). Furthermore, since the point U also lies on the radical planes s_p, s_t, s_{qr} of the pairs of complementary spheres of intersection (I_p), (I_p'); (I_t), (I_t'); (I_{qr}), (I_{qr}'), (U) is also orthogonal to the latter six spheres, for similar reasons. U is again seen to be on the radical planes m_a and m_d of the pairs of spheres (M), (A); (M), (D) (Art. 3a), therefore (U) is orthogonal to (A) as well as to (D) too.

Similarly for the other sphere of anti-similitude (U') of the two spheres (B'), (C').

Analogous considerations apply to the spheres of anti-similitude of the spheres (A'), (B'), (C'), (D') taken in pairs other than those considered above. Thus: *Given four intersecting spheres, each sphere of anti-similitude of the associated tetrad of spheres is orthogonal to eight of the spheres of intersections and to two of the given spheres.*

c. Simple calculations disclose that: *Given four intersecting spheres, each of their sixteen spheres of intersection is orthogonal to six of the spheres of anti-similitude of their tetrad of associated spheres: viz. to those whose centres lie on the radical plane of the sphere of intersection considered and its complementary one. The latter sphere is orthogonal to the same six spheres of anti-similitude. Each of the four given spheres is orthogonal to six of the spheres of anti-similitude under consideration, viz. to those whose centres lie on the radical plane of the sphere considered and their orthogonal sphere.*

9. Orthogonal Spheres. a. *The radical tetrahedron of four intersecting spheres coincides with the central tetrahedron of these spheres, if, and only if, the four given spheres are real and mutually orthogonal,*

in which case the central tetrahedron is orthocentric and self-polar [1] *for their fifth orthogonal sphere* (M).

b. It follows that: *Four mutually orthogonal real spheres coincide with their associated tetrad of spheres* (Art. 7a). *Therefore the eight planes of perspectivity of their pairs of complementary tetrahedra of intersection are their planes of similitude* [5] (Art. 7b), *and form two tetrahedra desmic with their central tetrahedron* [5] (Art. 4c); *the radical plane of the circumsphere of their central tetrahedron and their fifth orthogonal sphere coincides with their Newtonian plane* (Art. 7c); *each sphere of anti-similitude of theirs is orthogonal to eight of their spheres of intersection and to two of them* (Art. 8b); *each of their sixteen spheres of intersection is orthogonal to six of their spheres of anti-similitude; viz. to those whose centres lie on the radical plane of the sphere of intersection considered and its complementary one, the latter sphere is orthogonal to the same six spheres of anti-smilitude*[5], *each of the four given mutually orthogonal real spheres is orthogonal to six of their spheres of anti-similitude whose centres lie on the radical plane of the sphere considered and their fifth orthogonal sphere* (Art. 8c.)

c. U, U'; X, X' being pairs of conjugate points for (M) (Art. 3d), on the pair of opposite edges $B'C'$, $A'D'$ of (T') (Art. 3b), i.e. on the pair of polar edges BC, AD of the self-polar tetrahedron T for (M) (Art. 9a), $UU'XX'$ forms a tetrahedron self-polar for (M). Similarly are the other two tetrahedra $VV'YY'$, $WW'ZZ'$ of the desmic system (S) (Art. 4f) self-polar for (M). Therefore the desmic system (S''') (Art. 6c) coincides with (S), and thence their conjugates (S'') and (S') coincide making (S') too self-polar for (M).

Thus: *The two conjugate desmic systems of intersection of four mutually orthogonal and real spheres are each self-polar for their fifth orthogonal sphere.*

10. Umbilical Projection. a. Let us consider a tetrad of primes in a four dimensional space meeting in a point M, such that the lines common to the triads of them meet a given quadric variety (Q) in the pairs of points P, P'; Q, Q'; R, R'; T, T'. The polar prime (M)

of M w.r.t. (Q) meets the given primes in planes that form a tetrahedron (T') reciprocal to that, say (T), formed by their poles w.r.t. (Q), for its section by the prime (M).

Taking one point from each of the four pairs of points of intersection enumerated just above, we obtain eight pairs of perspective tetrahedra, enunciated in Art. 2a, whose planes of perspectivity are the planes common to their respective primes and lie in the prime (M) by an argument of Art. 3a giving rise to the configuration therein described in Art. 4. Therefore each of these eight pairs of primes as well as their sections with (Q) are harmonically inverse [9] w.r.t. (M), and their poles w.r.t. (Q) are collinear with M. The second centre of inversion [9] (harmonic) for the sections with (Q) of a pair of such primes is the pole of their common plane w.r.t. its section by (M).

The polar primes of the vertices of the tetrahedron (T') w.r.t. (Q) pass through M, and form an associated tetrad of primes in regard to the given tetrad of primes.

On projection [9] from an *umbilicus* of (Q) into a given solid we get all the results, obtained above, at once, including the special ones of the preceding articles when the given tetrad of primes together with (M) form a self-polar simplex for (Q). For, by this projection, prime sections of (Q) project into spheres whose orthogonality is implied by the conjugacy of the corresponding primes w.r.t. (Q); their poles w.r.t. (Q) project into the centres of the corresponding spheres; cross ratios and in particular the harmonic relations are unaltered; harmonic inversion w.r.t. a point and a prime [10] gives us inversion w.r.t. the corresponding sphere; the vertices of the hypercones through a pair of prime sections of (Q) project into the centres of similitude of the corresponding spheres; and so on.

b. Applying this type of projection [8] from an umbilicus of a quadric in space into a plane we get immediately the results obtained by Court [4] in a similar manner.

11. Isodynamic Tetrahedra, a. It may happen that the central and the radical tetrahedra (T), (T') (§ 1b) of the four given spheres are in perspective such that their plane of perspectivity coincides with that, say s (§ 2b), of a pair of complementary tetrahedra of intersection $I = PQRT, I' = P'Q'R'T'$. That is, the 6 pairs of corresponding edges of (T), (T') meet in the 6 points in s (§ 3b), e.g. $(BC,B'C') = U = (QR,Q'R')$. Now the 3 spheres (U), (B) and (C), having their centres U, B, C collinear and being orthogonal to (§ 8b) the same sphere (M), are coaxal. Again (U) being orthogonal (§ 8b) to the other two given spheres (A), (D) and to the pair of complementary spheres of intersection (I) and (I'), the two pairs of their common points Q, R; Q',R' (Table 1) are two pairs of inverse points for it. Thus (B), passing through R and R', inverts into (C), passing through Q and Q', w.r.t. it. Or, it is a sphere of anti-similitude for (B), (C) and passes through the two pairs of their common points P, T ; P', T' (Table 1) which lie on (I), (I') respectively.

Hence UP, UT are tangents to (I) at P, T and UP', UT' to (I') at P', T'. Or, (U) cuts the planes of the triangles PQR, QRT, $P'Q'R'$, $Q'R'T'$ in Apollonian circles [2 ; 5] with common centre at U.

Similarly behave the other five spheres (V), ... , (Z) (§ 8a). Thus I, I' are isodynamic [2] with (U),...,(Z) as their common Neuberg spheres and s as their common Lemoine plane [5 ; 12]. Hence:

If the central and the radical tetrahedra of four intersecting spheres be in perspective such that the plane of perspectivity s coincides with that of a pair of complementary tetrahedra I, I′ of intersection, s is a plane of similitude of the given spheres; I, I′ are both isodynamic such that their Lemoine planes coincide with s and the six spheres of anti-similitude of the four spheres, with centres in s, are their common Neuberg spheres. Their common 'Brocard [5] *diameter' meets their* 8 *altitudes and is perpendicular to s.*

b. The pair of tetrahedra like I, I' is called *semi-inverse* for the sphere (M) by Court[3a] who proves that: *I′ is the transform*

of I in the homology (M, s,—c) with centre of homology at the centre M of (M), the plane of homology being the radical plane s of its circumsphere (I) and (M), and the constant cross ratio of homology being equal to the negative ratio —c of the power of M for (I) to the square of the radius of (M). Obviously then (I) transforms into the circumsphere (I') of I' and the *tangential tetrahedron,* of I, formed of the tangent planes of (I) at the vertices of I, into that of I' in this homology. Hence *An isodynamic tetrahedron transforms into a like tetradedron in the said homology such that their Lemoine planes are coaxal with the plane of homology.*

c. In fact, *the circumsphere of an isodynamic tetrahedron is the 'polar quadric' or 'second polar' of its Lemoine point w.r.t. it* in analogy with the polar conic of a point w.r.t. a triangle [13-17]. For, *the tangential tetrahedron of an isodynamic tetrahedron is its 'anticevian* [11] *tetrahedron' for its Lemoine point whose polar for its circumsphere coincides* [2] *with that for the two tetrahedra as its Lemoine plane.*

d. Thus we may observe that: *If a tetrahedron of intersection of four intersecting spheres be isodynamic, similarly behaves its complementary one such that their Lemoine planes k, k' are coaxal with their plane of perspectivity s. Further if k' coincides with k, s too coincides with k such that the central and the radical tetrahedra of the given spheres are in perspective having the same s as their plane of perspectivity.*

The later part of the proposition follows by retracing the steps of § (11a) and recalling that s is the radial plane of the corresponding pair of complementary spheres of intersection (§ 5c).

e. As an immediate consequence now it follows that: *If two non-complementary tetrahedra of intersection for four intersecting spheres be isodynamic such that their Lemoine planes coincide respectively with those of their complementary ones, the same is true of the other six pairs of complementary tetrahedra of intersection, and the four spheres are orthogonal* [5].

For the central and the radical tatrahedra of the four spheres are now, by § 11d, doubly perspective in the same order, but such a situation can never occur and therefore they coincide (§ 9a).

f. A triangle is inherently isodynamic and thus leads to the *Lemoine Geometry* [3b]. Hence follows the following theorem of some significant interest as a contrast to the preceding two propositions:

If the Lemoine axes of a pair of complementary triangles of intersection for three intersecting circles in a plane coincide, the same is true of the other three such pairs of them and the given circles are mutually orthogonal [3c].

Reference may also be made to the recent paper of court [5 a] who himself has developed similar ideas, independently of course, in his masterly style.

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Indian Inst. of Tech.
Kharagpur

UNI-AND DEMI-ORTHOCENTRIC SIMPLEXES

By SAHIB RAM MANDAN

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The study of the altitudes of a simplex in an n-space grows more and more complex as n increases, nevertheless it forms an interesting theme and thus makes it worthwhile to have some glimpses of these new unseen highlands. The present paper is a fourth one in the series of them on the same topic, the former three being in press for which, however, reference can be made to their '*abstracts*' published in the '*Proceedings of the 45th (1958), 46th(1959) and 47th(1960) Session of the Indian Science Congress Association*', one after the other. Here we take up only two special types of simplexes and observe how certain special points and special hyperspheres associated with them are related to one another. The main specialities are as follows:

(i) A simplex $S_{r.(n-r)}$ is said to be UNI-ORTHOCENTRIC (UoS) when its r altitudes concur with its *special* r*-altitude* l at its r*-orthocentre* H and other $n-r$ with its special $(n-r)$-altitude l' at its $(n-r)$-orthocentre H'. l,l' and its $(n+1)$th altitude, lie in a plane, CONCURRENT at its uni-orthocentre U. Evidently, to meet such simplexes, we must rise and peep into spaces higher than a solid. The *S-point* (Monge point) of the UoS lies at the centroid of U,H,H' for *multiples* 1, $r-1$, $n-r-1$ respectively. The developments of an UoS $Sq.\ r.\ (n-q-r)$ in 6-space onwards and $Sp.\ q.\ r.\ (n-p-q-r)$ in 8-space onwards, etc. are suggested.

(ii) A simplex $Sq.\,r.\,(n-q-r+1)$ is said to be DEMI-ORTHOCENTRIC (DoS) when its q altitudes concur with its special q-altitude l' at its q-orthocentre H', other r with its special r-altitude l at its r-orthocentre H and the rest with its special $(n-q-r+1)$-altitude l'' at its $(n-q-r+1)$-orthocentre H''. l,l',l'' then CONCUR, at its DI-ORTHOCENTRE D, as the 3 altitudes of the triangle, formed of their feet in their relative $(r-1)$, $(q-1)$ and $(n-q-r)$ faces of the DoS, perspective to that formed by its 3 *Demi-orthocentres*, viz. $HH'H''$, from D as the centre of perspectivity. Obviously, to treat such simplexes, we must enter a 5-space onwards. The S-point of the DoS lies at the centroid of D,H,H',H'' for multiples 1, $r-1$, $q-1$, $n-q-r$ respectively. The developments of a DoS $Sp.\ q.r.\ (n-p-q-r+1)$ in 7-space onwards and $Sp.\ q.r.s.\ (n-p-q-r-s+1)$ in 9-space onwards, etc. are suggested.

The knowledge of the propositions of incidence and perpendicularity in an n-space is assumed.

INTRODUCTION

1. A special simplex $S_r(A)$. The common perpendicular secant of an $(r-1)$-space of a simplex $S(A)=A_0,\ldots,A_n$ and its opposite $(n-r)$-space is said to be its r or $(n-r+1)$-*altitude* l

(*bialtitude* for $r=2$) to them. When the altitudes of $S(A)$, from its r vertices in the $(r-1)$-space, concur at H say, H lies [3] on l and the $(r-1)$-space is perpendicular to the $(n-r)$-space. $S(A)$ is then said to become $S_r(A)$, the $(r-1)$-space and its relative r-altitude to be *special*, and H as its *r-orthocentre* (*biorthocentre* for $r=2$). Evidently : 'Every $(p-1)$-space of the $S_r(A)$ in its special $(r-1)$ space is special ; all its relative p-altitudes are special too and pass through H. In particular, all its edges therein are special and perpendicular [3] to its respective opposite $(n-2)$-spaces ; all its relative bialtitudes are special too and pass through H.'

2. The S-point of a simplex. 2.1. We define the S-point of a simplex $S(A)$ in an n-space by the following proposition: 'The normals to the prime faces of the $S(A)$ at the points, which divide the join of their S-points and the feet of its respective altitudes in the ratio $1:(n-2)$, CONCUR at its *S-point S*'. In establishing [3] this theorem, we also prove there that : 'S lies at its Monge point [1] on its *Euler line* (the join of its circumcentre O and centroid G) such that G divides SO in the ratio $2:(n-1)$.'

2.2. It is further shown there [3] that: 'The S-point of a SEMI-ORTHOCENTRIC simplex (SoS) $S_{r(n-r+1)}$ lies on the join of its r- & $(n-r+1)$-orthocentres (*principal* r-altitude) dividing the same in the ratio $(n-r):(r-1)$.'

2.3. Again we see [5] that : 'All the 5 tetrahedral faces of an UoS $S_{2.2}$ are semi-orthocentric and its S-point lies at the centroid of the triangle formed by its uni-orthocentre and 2 bi-orthocentres.'

I UNI-ORTHOCENTRIC SIMPLEX (UoS).

3. The Uni-orthocentre of an UoS. Let the r altitudes from the vertices $A_1, \dots, A_r$ of a simplex $S(A)$ concur at H, and $n-r$ from $A_{r+1}, \dots, A_n$ at H'. $S(A)$ is then said to become an $S_{r.(n-r)}$. If A be the projection of H in its special $(r-1)$-space $x = A_1, \dots, A_r$ and A' of H' in its special $(n-r-1)$-space $y = A_{r+1}, \dots, A_n$, AH, $A'H'$ are then its relative special r- & $(n-r)$-altitudes and therefore both perpendicular to both x, y (Art. 1). Consequently, $A_0, \dots, A_r$

$A'H'$ is the $(r+1)$-space normal to y at A' and AH lies in it; $A_0 A_{r+1} \ldots A_n AH$ is the $(n-r+1)$-space normal to x at A and $A'H'$ lies in it. Thus AH, $A'H'$ lie in the common perpendicular plane $A'AA_0$ of x, y as the 2 altitudes of the triangle $AA'A_0$ meeting in its orthocentre U say. For $A'A_0$, AA_0 lie respectively in the $(n-r)$- & r-spaces of the simplex relative to AH, $A'H'$. The third altitude UA_0 of this triangle is then seen to be no other than the altitude of the simplex too from A_0. The unique position that U occupies in regard to the simplex suggests the name '*uni-orthocentric*' for it and its '*uni-orthocentre*' for U. Hence: *If A, A' be the respective feet of the special r- & $(n-r)$-altitudes l, l' of an UoS $S_{r.(n-r)}$ in its relative special $(r-1)$- & $(n-r-1)$-spaces, its altitude from its $(n+1)$th vertex A_0, l, l' CONCUR as the 3 altitudes of the triangle $AA'A_0$.*

4. The Prime Faces of an ***UoS***. Let U_i, H_i, H'_i be the respective projections of U, H, H' in the prime face a_i opposite a vertex A_i of the *UoS* of the preceding para. (i) a_0 is evidently semi-orthocentric with H_0, H'_0 as its r- & $(n-r)$-orthocentres on its principal r-altitude AA' and U_0 is the foot thereon of the altitude of the *UoS* to it. It is therefore said to be the *special prime* of the *UoS*.

(ii) If $r=2$, a_i $(i=1, 2)$ is also semi-orthocentric with H_i as its biorthocentre and H'_i as its $(n-2)$-orthocentre on its principal bialtitude $A'U_i$, and H_i is the foot of the altitude of the *UoS* to it on its altitude, from A_j $(j=1, 2, i\neq j)$, which is the projection therein of the special bialtitude AUH of the *UoS* to A_iA_j, its altitude A_0U_i being that of the altitude A_0U of the *UoS*.

(iii) If $r=n-2$, a_i $(i=n-1, n)$ is also semi-orthocentric with H_i as its r-orthocentre and U_i as its biorthocentre on its principal bialtitude AU_i, and H'_i is the foot of the altitude of the *UoS* to it on its altitude from A_j $(j=n-1, n, i\neq j)$.

(iv) If $2<r<n-2$, a_i $(i\neq 0)$ is *uni-orthocentric*; H_i, H'_i are its $(r-1)$- & $(n-r)$-orthocentres for $i=1, \ldots, r$ and r- & $(n-r-1)$-orthocentres for $i=r+1, \ldots, n$; U_i is its *uni-orthocentre*. For, the altitude A_0U of the *UoS* projects into its altitude A_0U_i, the r-altitude AHU of the *UoS* into its r-altitude AH_iU_i $(i=r+1, \ldots, n)$

and the $(n-r)$-altitude $A'H'U$ of the UoS into its $(n-r)$-altitude $A'H'_iU_i$ $(i = 1, \dots, r)$.

Thus : *The uni-orthocentre of an UoS projects into that of its every uniorthocentric prime face, and into the biorthocentre of its each semi-orthocentric one other than its special prime. The principal r-altitude of a semi-orthocentric prime face of a simplex meets its altitude to this face, if, and only if, it becomes an UoS.*

5. The S-point of an UoS. Let us assume that the S-point S_i of an *uni-orthocentric* prime face a_i $(i = 1, \dots, r \neq 2)$ of the UoS just considered is the centroid of U_i, H_i, H'_i for multiples 1, $r-2$, $n-r-1$. That is, if P_i be the point on H'_iU_i dividing it in the ratio $1 : (n-r-1)$, S_i lies on H_iP_i dividing it in the ratio $(n-r) : (r-2)$.

Now if the normal to a_i at P_i meets $H'U$ in P, that at the point T_i, on H_iP_i dividing it in the ratio $(n-r) : (r-1)$ meets HP in the S-point S (Art. 2.1) of the UoS. For, P is a fixed point on $H'U$ dividing it in the ratio $1 : (n-r-1)$ and therefore S too on HP dividing it in the ratio $(n-r) : (r-1)$; H_i is evidently the foot of the altitude of the UoS to a_i, and T_i is seen to lie on S_iH_i dividing it in the ratio $1 : (n-2)$. Thus : *The S-point of an UoS* $S_{r.(n-r)}$ *is the centroid of its uni-, r- & $(n-r)$-orthocentres for multiples* 1, $r-1$, $n-r-1$. But its truth is based on that of our assumption which is true in a 4-space (Art. 2.3), and hence by induction it follows. It can also be confirmed by constructing S by means of the S-points of the semi-orthocentric prime faces of the UoS (Arts. 4, 2.2).

6. The F- & M-points of an UoS. *The normals to the prime faces of an UoS* $S_{r\cdot(n-r)}$ *opposite its vertices in its special $(r-1)$-face at their respective S-points (centroids) CONCUR at its 'M-point M (F-point F)' and those to its other prime faces, excepting its special prime a_0, CONCUR at its second M-point M' (F-point F'). M is the centroid of its uni-, r- & $(n-r)$-orthocentres U, H, H' for multiples* 1, $r-2$, $n-r-1$, *and M' for multiples* 1, $r-1$, $n-r-2$; *F, F' lie respectively on the joins of its circumcentre O to M, M' dividing*

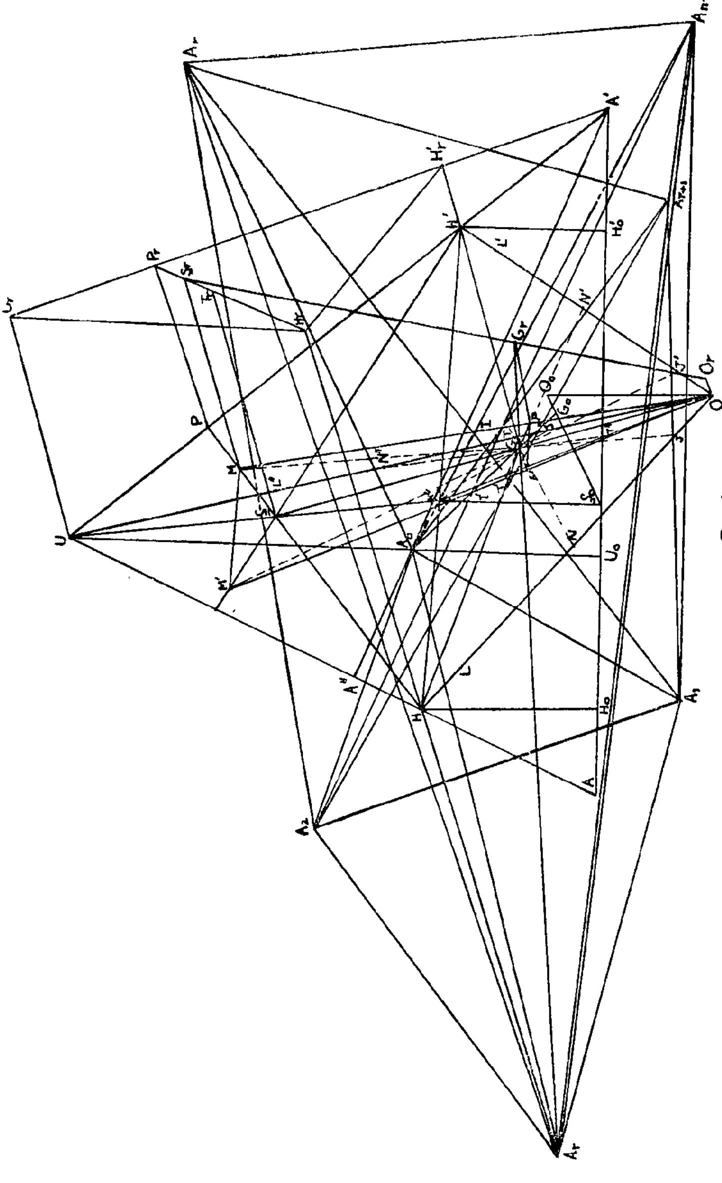

Fig. 1

them in the ratio $(n-2):2$. *Thus* : *M lies on* $H'U$ *dividing it in the ratio* $1:(n-3)$ *when* $r=2$, *and* M' *on* HU *dividing it in the same ratio when* $r=n-2$ (Fig. 1).

It follows immediately from the preceding proposition and the definition of the S-point as related to *the circumcentre* O and the centroid G of a simplex (Art. 2.1). Hence : *F, F' lie respectively on GH, $G'H'$ dividing them in the ratio* $1:(n+1)$ *externally, FF' is parallel to both HH', MM' and lies in the plane, parallel to their plane UHH', through the centroid G_0 of a_0.*

7. The Orthocentres of the Triangles and Tetrahedra of an *UoS*.

7.1. *The tetrahedra, of an UoS* $S_{r.(n-r)}$ *having just one edge in its special* $(r-1)$*-space* x *or* $(n-r-1)$*-space* y, *are all evidently semi-orthocentric, and the rest are all orthogonal whose orthocentres are the orthogonal projections of its r-orthocentre* H *or* $(n-r)$*-orthocentre* H' *according as 3 vertices of one lie in* x *or* y. *The semi-orthocentre of a semi-orthocentric tetrahedron relative to its either special edge is the projection, of* H *or* H' *according as the edge lies in* x *or* y, *and of the uni-orthocentre* U *of the UoS otherwise.*

Let the *UoS* be as before (Art. 3), and B be the semi-orthocentre of a tetrahedron $T=A_nA_0A_1A_2$ relative to its special edge A_nA_0 which lies neither in x nor in y. That is A_0B, A_nB are 2 altitudes of T. A_0U being an altitude of *UoS*, the plane BA_0U and therefore BU are perpendicular to the plane $A_nA_1A_2$.

Again BU is easily seen to lie in the $(n-2)$-space, normal to the plane $A_0A_1A_2$, through the $(n-r)$-space $A_{r+1}, \ldots, A_nA'H'$. Therefore BU is perpendicular to the solid T at B. That proves the later part of the proposition, the former can also be shown [3] by a similar argument. Similarly it follows that :

The orthocentre of a triangle of the UoS is the orthogonal projection, of H *or* H' *according as it has an edge in* x *or* y, *and of* U *otherwise.*

7.2. *A simplex* $S(A)$ *in an n-space becomes an UoS* $S_{r.(n-r)}$, *if, and only if, a pair of its opposite r-and* $(n-r)$*-faces* $A_0, \ldots, A_r$, $A_{r+1}A_nA_0$ *are both orthogonal and the* $(n-2)$*-spaces normal to its*

$r(n-r)$ triangular faces $A_i A_0 A_j (i=1,\ldots,r; j=r+1,\ldots,n)$ at their respective orthocentres CONCUR at its uni-orthocentre U.

The $(n-2)$-space normal to a triangular face $A_0A_1A_j$ at its orthocentre and its altitude from A_j determine the hyperplane, normal to A_0A_1 from U at A_{01} say, which then contains the altitude $A_{01}A_j$ of every such triangle for all values of j and therefore the $(n-r-1)$-space $y=A_{r+1}\ldots A_n$. Thus A_0A_i for every value of i and therefore the r-space $A_0\ldots A_r$ are perpendicular to y. Every edge of $S(A)$ in y is then perpendicular to its opposite $(n-2)$-space and therefore is special (Art. 1), for the $(n-r)$-simplex $A_{r+1}\ldots A_nA_0$ is given to be orthogonal. Similarly its every edge in the $(r-1)$-space $x=A_1\ldots A_r$ is special and hence the proposition.

8. The N-, J-, L-, & H-**hyperspheres of an** *UoS*. 8.1. *The 9-point, g-, l-and polar circles of the triangular faces of an UoS $S_{r.(n-r)}$ through its edges in its special $(r-1)$-space x lie respectively on an N-, J-, L- & H-hypersphere. The same is true for these circles of its triangular x faces through its edges in its special $(n-r-1)$-space y, and the rest of the $r(n-r)$ circles of each type lie respectively on a third N-, J-, L- and H-hypersphere.*

There is a family of 5 coaxal circles *associated*[2] with a triangle, viz. its circumcircle and 9-point circle, their 2 circles of antisimilitude (one being its polar circle with centre at its orthocentre, the other with centre at its centroid is called its *g-circle*) and their circle of similitude (called its *l-circle*) ; their centres lie on its *Euler line* such that its centroid (9-point centre) trisects (bisects) the segment between its circumcentre and orthocentre.

Now there are 3 types of triangles of the *UoS* according as one has an edge in x, in y, or in neither. The $(n-2)$-spaces normal to the triangular faces of the *UoS* at their respective circumcentres all concur at its circumcentre O ; those at their orthocentres concur at its r-orthocentre H, $(n-r)$-orthocentre H', or uni-orthocentre U according as the faces belong to first, second or third type (Art. 7). Therefore those at their centroids (9-point centres) concur at the trisection points J, J', J'' (midpoints N, N', N'') of OH, OH', OU respectively.

Thus their 9-point circles lie on its 3 N-hyperspheres (N), (N'), (N'') with centres at N, N', N''; their g-circles on its 3 J-hyperspheres (J),(J'), (J'') with centres at J, J', J''; their polar circles lie on its 3 H-hyperspperes (H), (H'), (U) with centres at H, H', U; their l-circles lie on its 3 L-hyperspheres (L), (L'), (L'') with centres at the midpoints L, L', L'' of JH, $J'H'$, $J''U$. Hence :

The N-, J-, L- & H-hyperspheres of the UoS distribute into 3 *coaxal families of four each, viz.* f : (H), (L), (N), (J) ; f' : (H'), (L'), (N'), (J') ; f'' : (U), (L''), (N''), (J''), *such that its circumhypersphere* (O) *belongs to them as their common fifth one.*

8.2. *The* $(n-3)$*-sphere common to the* 3 *N-hyperspheres of the UoS lies in the 'mediator'* $(n-2)$*-space (normal bisector) m of the principal r-altitude of its special prime* a_0.

For the midpoints of the non-special edges $A_i A_j$ (Art. 7.2) of a_0 lie on the said sphere which is its principal[3] $(n-3)$-sphere.

8.3. *The* $(n-3)$*-sphere common to the* 3 *H-hyperspheres of the UoS also lies in m.* From the property of the polar circle of a triangle and the construction of the H-hyperspheres, it is apparent that the special $(r-1)$-space x of the *UoS* and its opposite $(n-r)$-space are polar for both (H), (U), and are both perpendicular to their line of centres UH meeting the same therefore in the pair of their limiting points (common inverse points) A, A'' say, which are then the feet of its special r-altitude therein. Thus the radical hyperplane of (H), (U) coincides with the mediator, of AA'', which obviously passes through m. Similarly, the radical prime of (H'), (U) also passes through m, and hence the result.

8.4. Thus : *The* $(n-4)$*-sphere common to the* 3 *families* f, f', f'' *associated with the UoS lies in* m.

8.5. It can be proved[2] that : *The normal to a_0 at its S-point, the join of U to the S-point of the UoS and that of J'' to its centroid G CONCUR on HH'* at M'' *(say), called its third M-point, such that the normal to a_0 at its centroid, UG, OM'' CONCUR at F'' (say), called the third F-point of the UoS. The triangles $JJ'J''$, $MM'M''$* (Art. 6) *are homothetic w. r. t. G, homothetic ratio being* $(n-2):3$ (Fig. 1).

9. The M-, F-, I-& T-hyperspheres of an UoS. 9.1. There is a family of 5 coaxal hyperspheres associated [3] with a simplex in an n-space, viz. its circumhypersphere (O) and $3(n+1)$ *pointsphere*, their 2 hyperspheres of *antisimilitude* (one being its *quasi-polar* one (S) with centre at its S-point S such that the power of S w.r.t. (O) is n times the square of the radius of (S), the other with centre at its centroid is called its *G-sphere*) and their hypersphere of *similitude* (called its *GS-sphere*). We have seen there that : ' The quasi-polar $(n-2)$-spheres and G-, GS-& $3n$-point-spheres of the 2 prime faces opposite the vertices comprising the special edge of an $S_2(A)$ in an n-space (Art. 1) lie respectively on an M-, F-, I- & T-hypersphere with centre at its M-, F-, I-, & T-point M, F, I, T (Art. 6) such that $MI = IF$, MT: $TF = n$: $(n-2)$'. Thus it follows that : *The quasi-polar* $(n-2)$*-spheres,* G-, GS-& $3n$*-point-spheres of the* r *prime faces opposite the vertices of an* UoS *in its special* $(r-1)$*-space lie respectively on an* M-, F-, I-& T*-hypersphere belonging to a coaxal family to which belongs its circumhypersphere ; all others, except those in its special prime, behave alike.*

9.2. Now n points of the $3n$-point-sphere of a prime face a of the UoS are the centroids of its $n(n-2)$-faces therein opposite its n edges through its vertex opposite a, and are therefore homothetic to their n mid-points w. r. t. its centroid G, homothetic ratio being 2: $(n-1)$. Thus : *The centroids of the* $n+r(n-r)$ $(n-2)$*-faces of the* UoS *opposite its non-special edges lie on an hypersphere, homothetic to its* N*-hypersphere* (N'') w. r. t. G (Art. 8.1), *referred as its third* T*-hypersphere* (T''), *its former two* (T), (T') *to* (N), (N'), *such that the* $3n$*-point-sphere of its special prime face* a_0 *lies on* (T''). Hence : *The centroids of its* $r(n-r)$ $(n-2)$*-faces opposite its non-special edges in* a_0 *lie on an* $(n-3)$*-sphere, common to its* 3 T*-hyperspheres, homothetic to that common to its* 3 N*-hyperspheres w. r. t.* G.

9.3. *The centres of the* T*-hyperspheres of the* UoS *are homothetic to its* M*-points as well as to its* F*-points* (Arts. 6, 8.5) *w. r. t. its circumcentre with homothetic ratios* $(n-2)$: $(n-1)$, n : $(n-1)$.

10. The other uni-orthocentric simplexes 10.1. Let the q altitudes from the vertices $A_1, \ldots, A_q$ of $S(A)$ (Art. 1) concur with its relative special q-altitude l at its q-orthocentre H, r from $A_{q+1,\ldots,} A_{q+r}$ with its relative special r-altitude l' at H', $n-q-r$ from $A_{q+r+1,\ldots,} A_n$ with its relative special $(n-q-r)$-altitude l'' at H''; l, l', l'' meet its relative special $(q-1)$-, $(r-1)$- & $(n-q-r-1)$-spaces y, z, t in A, A', A'' respectively. Evidently, $T = AA'A''A_0$ is the common perpendicular solid of y, z, t. $l = AH$, $l' = A'H'$, $l'' = A''H''$ are then seen to be 3 altitudes of the tetrahedron T, the fourth coinciding with the $(n+1)$ th one of $S(A)$. If T is orthogonal, $S(A)$ is said to become an UoS $S_{q.r.(n-q-r)}$ with its uni-orthocentre at the orthocentre of T. Thus: *If the $q-1$ $(q<n-1)$ consecutive edges of a q-gon formed of q vertices of a simplex $S(A)$ in an n-space be special, all its edges in their $(q-1)$-spaces y are special and it then becomes $S_q(A)$* [3] ; *if the $r-1$ $(r<n-r+1)$ consecutive edges of an r-gon formed of its other r vertices be also special, all its edges in their $(r-1)$-space z are special and the $(q+r)$-simplex formed of its vertices in y, z and one other vertex then becomes an UoS* $S_{q.r}$ (Art. 3); *if the $n-q-r-1$ consecutive edges of an $(n-q-r+1)$-gon formed of the remaining vertices of $S(A)$ be special too, all its edges in their $(n-q-r-1)$-space t are special and it becomes an UoS $S_{q.r.(n-q-r)}$, if, and only if, its $(n-r-1)$-space yt and $(n-q-1)$-space zt are perpendicular to the respective opposite r — & q-spaces, in which case its $(q+r-1)$-space yz is also perpendicular to its opposite $(n-q-r)$-space such that its q-, r-, $(n-q-r)$-, $(q+r)$-, $(n-r)$- & $(n-q)$-altitudes relative respectively to y, z, t, yz, yt, zt CONCUR with its altitude to its 'special prime' yzt at its uni-orthocentre.*

10.2. Similarly, *if $p-1$, $q-1$, $r-1$, $n-p-q-r-1$ consecutive edges, of an $(n+1)$-gon formed of the vertices of $S(A)$, having no vertex common, be special and three of the 4 joins of their $(p-1)$-, $(q-1)$-, $(r-1)$- & $(n-p-q-r-1)$-spaces x, y, z, t to its $(n+1)$th vertex A_0 are perpendicular to the respective opposite $(n-p-1)$-, $(n-q-1)$-, $(n-r-1)$-, & $(p+q+r-1)$-spaces of $S(A)$, the fourth behaves alike and it becomes an UoS*

$S_{p.q.r.(n-p-q-r)}$ *such that its* p-, q-, r-, $(n-p-q-r)$-, $(q+r)$-, $(r+p)$-, $(p+q)$-, $(n-q-r)$-, $(n-r-p)$-, $(n-p-q)$-, $(n-p)$-, $(n-q)$-, $(n-r)$- & $(p+q+r)$- *altitudes relative respectively to* $x, y, z, t, yz, zx, xy, xt, yt, zt, yzt, ztx, txy, xyz$ *concur with its altitude to its 'special prime'* $xyzt$ *at its uni-orthocentre* U *as the* 5 *altitudes and* 10 *bialtitudes of the orthogonal* 4*-simplex, formed by* A_0 *and the* 4 *feet of its special* p-, q-, r- & $(n-p-q-r)$-*altitudes in* x, y, z, t *respectively, perspective with the one formed of the foot of* AoU *and its* p-, q-, r- & $(n-p-q-r)$-*orthocentres from* U *as their centre of perspectivity.*

Similarly we can define an *UoS* $S_{p.q.r...(n-p-q-r-...)}$ and observe that its n' special p-, q-, r-, ...-& $(n-p-q-r...)$-altitudes *concur* with its $(n+1)$th altitude at its *uni-orthocentre* U as the altitudes of the *associated* orthogonal n'-simplex, formed by its $(n+1)$th vertex A_0 and their n' feet in its respectively relative n' special $(p-1)$-, $(q-1)$-, $(r-1)$-,... - & $(n-p-q-r-...-1)$-spaces, perspective with the one formed of the feet of A_0U and its n' p-, q-, r-,...& $(n-p-q-r)$-orthocentres from U as their centre of perspectivity.

10.3. On the strength of certain observations made below (Art. 12.2) in regard to a special *DoS* $S_{2.r.(n-r-1)}$ we can also establish the following proposition by the method of induction :

The S-point of an UoS $S_{p.q.r....(n-p-q-r-...)}$ *lies at the centroid of its uni-orthocentre and* $n'p$-, q-, r-, ... -& $(n-p-q-r-...)$-*ortho-centres for multiples* $n'-1, p-1, q-1, r-1, ...,$ & $(n-p-q-r-...-1)$ *respectively.*

Thus the S-point of the *UoS* $S_{q.r.(n-q-r)}$, discussed just above, lies at the centroid of U, H, H', H'' for multiplies $2, q-1, r-1, n-q-r-1$; and that of $S_{p.q.r.(n-p-q-r)}$ at the centroid of its uni-orthocentre, p-, q-, r- & $(n-p-q-r)$-orthocentres for multiples 3, $p-1, q-1, r-1, n-p-q-r-1$ (cf. Art. 5).

II DEMI-ORTHOCENTRIC SIMPLEX (*DoS*).

11. The Demi-orthocentre of a *DoS.* Let the q altitudes from the vertices $A_0, \dots, A_{q-1}$ of $S(A)$ (Art. 1) concur at H, r from A_q ....

A_{q+r-1} at H' and the rest at H''. If A be the projection of H in its special $(q-1)$-space $y = A_0, \ldots, A_{q-1}$, A' of H' in its special $(r-1)$-space $z = A_q, \ldots, A_{q+r-1}$, A'' of H'' in its special $(n-q-r)$-space $t = A_{q+r} \ldots A_n$, AH, $A'H'$, $A''H''$ are its respectively relative special q-, r- & $(n-q-r+1)$-altitudes. They lie, therefore, in the common perpendicular plane $AA'A''$ of y, z, t as the 3 altitudes of the triangle $AA'A''$ and thus concur at its orthocentre D. $S(A)$ is then said to be *demi-orthocentric DoS* $S_{q.r.(n-q+1)}$ with D as its di-orthocentre and H, H', H'' as its 3 *demi-orthocentres*. Thus: *if* $q-1$, $r-1$, $n-q-r$ *consecutive edges, of a simplex in an n-space, having no vertex common, are special, it becomes a DoS.*

12. The S**-point of a** ***DoS***. 12.1. Let P_i be the projection of a point P in the prime face a_i opposite A_i of the *DoS* just considered. Consider the same for $q=2$. $a_i (i=0, 1)$ is then *uni*-orthocentric (Art. 3) with D_i as its *uni*-orthocentre and H_i as the foot of the altitude of the *DoS* to it on its altitude, from $A_j (j = 0, 1, i \neq j)$, which is the projection therein of the special bialtitude AHD of the *DoS* to $A_0 A_1$; H'_i is its r-orthocentre on its special r-altitude $A'D_i$, H''_i is its $(n-r-1)$-orthocentre on its special $(n-r-1)$-altitude $A''D_i$. Its S-point S being the centroid of D_i, H'_i, H''_i for multiples 1, $r-1$, $n-r-2$ (Art. 5), its normal at S_i meets the plane $DH'H''$ in a fixed point M, viz. the centroid of D, H', H'' for the same multiples. Now if T_i be the point on $S_i H_i$ dividing it in the ratio $1 : (n-2)$, the normal to a_i at T_i meets HM in its S-point S (Art. 2.1) which is then the centroid of D, H, H', H'' for multiples 1, 1, $r-1$, $n-r-2$ as desired.

12.2. The tetrahedron $T = A_0 A_1 A' A''$ is evidently semi-orthocentric with D, H as its 2 semi-orthocentres. It may happen that the r-space zA_0 (Art. 11) is perpendicular to the opposite $(n-r-1)$-space tA_1. That is, a second pair of opposite edges $A_0 A'$, $A_1 A''$ besides $A_0 A_1$, $A'A''$ of T are rectangular, and then the third pair too behave alike. Consequently the r-space tA_0 is perpendicular to the opposite $(n-r-1)$-space zA_1 of the *DoS*, T is orthogonal, and D, H coincide at its orthocentre. Such a *DoS* may be said to be a *special* one, and it follows that : *The S-point of a special DoS* $S_{2.r.(n-r-1)}$ *lies at the centroid of its di-, r- &* $(n-r-1)$-

orthocentres for multiples 2, $r-1$, $n-r-2$. For its *bi-* & *di*-orthocentres coincide.

12.3. For $q>2$, $a_i(i=0,\ldots,q-1)$ is evidently demi-orthocentric with D_i, H_i, H_i', H_i'' as its *di*-, $(q-)$, r- & $(n-q-r+1)$-orthocentres. Now it follows by induction that : *The S-point of a DoS* $S_{q.r.(n-q-r+1)}$ *lies at the centroid of its di-, q-, r- &* $(n-q-r+1)$*-orthocentres for multiples* 1, $q-1$, $r-1$, $n-q-r$.

13. The F**-&** M**-points of a** *DoS*. *The normals to the prime faces, opposite the vertices in the special* $(q-1)$-, $(r-1)$- & $(n-q-r)$-*spaces of a DoS* $S_{q.r.(n-q-r+1)}$, *at their respective S-points (centroids) concur respectively at its M-points* M, M', M'' (*F-points* F, F', F'') *which lie at the centroids of its di-, q-, r-&* $(n-q-r+1)$-*orthocentres* D, H, H', H'', *for multiples* $1, q-2, r-1, n-q-r$; $1, q-1, r-2, n-q-r$; $1, q-1, r-1, n-q-r-1$; F, F', F'' *lie respectively on the joins of its circumcentres* O *to* M, M', M'' *dividing them in the ratio* $(n-2):2$ *as well as on the joins of its centroid* G *to* H, H', H'' *too dividing them in the ratio* $1:(n+1)$ *externally* (Fig. 2). Thus : *The triangle* $FF'F''$ *has its sides parallel to those of* $HH'H''$, $MM'M''$ *both, and therefore lies in a plane parallel to the common one of the latter two triangles* (cf. Art. 6) *which are homothetic w. r. t. the S-point S of the DoS with homothetic ratio* $(n-2):1$ (Art. 12.1).

Now if we introduce a *fourth M-point* M''' on DS such that $M'''S:SD=1:(n-2)$, and *F-point* F''' on DG such that $GD: F'''G=n:1$, we find O, F''', N''' collinear such that $OF''':F'''M=(n-2):2$.

14. The Orthocentres of the Triangles and Tetrahedra of a *DoS*. *The orthocentre of a triangle of a DoS* $S_{q.r.(n-q-r+1)}$ *is the orthogonal projection of its* q-, r-, *or* $(n-q-r+1)$*-orthocentre* H, H' *or* H'' *according as it has an edge in its special* $(q-1)$-, $(r-1)$- *or* $(n-q-r)$*-space* y, z, t, *and of its di-orthocentre* D *otherwise.*

Let the *DoS* be as above (Art. 11) and C be the orthocentre of a triangle $A_iA_jA_k(i=0,\ldots,q-1; j=q,\ldots,q+r-1; k=q+r,\ldots,n)$. The hyper-planes normal to the sides of this triangle from C are

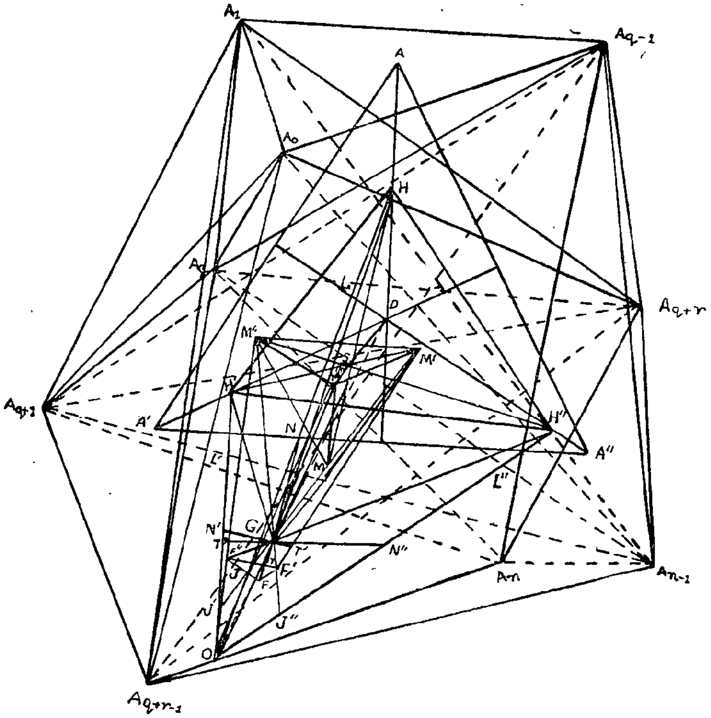

Fig 2

readily seen to contain D. That proves the latter part of the proposition and similarly follows the former. Hence: *The terahedra of the DoS having just one edge in y, z or t are all evidently semi-orthocentric, and the rest are all orthogonal whose orthocentres are the orthogonal projections of H, H', H'' according as 3 vertices of one lie in y, z or t. The semi-orthocentre of a semi-orthocentric tetrahedron relative to its either special edge is the projection of H, H', H'' according as the edge lies in y, z or t, and of D otherwise* (cf. Art. 7.1).

15. The Hyperspheres Associated with the ***DoS*****.** 15.1. *The polar, 9-point, g- & l-circles of the triangular faces of the DoS through its edges in y lie respectively on an H-, N-, J- & L-hypersphere belonging to a coaxal family to which belongs its circumhypersphere (O). The same is true for these circles of the triangular faces through*

its edges in z or t. The remaining q. r. $(n-q-r+1)$ *circles of each type too lie respectively on a fourth H-, N-, J- & L-hypersphere belonging to a fourth like family with centres on the join of its circumcentre O and di-orthocentre D.* (cf. Art. 8.1). It follows from the preceding proposition.

15.2. *The quasi-polar* $(n-2)$*-spheres* $(3n)$ *point-, GS- & G-spheres of the q primes opposite the vertices of the DoS in y lie respectively on an M-, T-, I- & F-hypersphere belonging to a coaxal family to which belongs (O). The same is true for these spheres of its r primes opposite its vertices in z. The remaining* $n-q-r+1$ *spheres of each type lie respectively on a third M-, T-, I- & F-hypersphere belonging to a third like family* (cf. Art. 9.1).

15.3. *The centroids of the* $qr+(q+r)(n-q-r+1)$ $(n-2)$*-faces opposite the non-special edges of the DoS lie on an hypersphere, homothetic to its fourth N-hypersphere w. r. t. its centroid G with homothetic ratio* -2 : $(n-1)$, *referred as its fourth T-hypersphere, the former three being homethetic to its first three N-hyperspheres* (Art. 9.2). Hence: *The joins of the centres of its corresponding N- & T-hyperspheres CONCUR at G and are divided thereat in the ratio* $(n-1):2$. *Those of its J-hyperspheres to its corresponding M-points* (Art. 12) *also CONCUR at G and are divided thereat in the ratio* $(n-2):3$ (cf. Art. 8.5).

16. The Other Demi-orthocentric simplexes. 16.1. *If* $p-1$, $q-1$, $r-1$, $n-p-q-r$ *consecutive edges, of an* $(n+1)$*-gon formed of the vertices of a simplex S(A) in an n-space, having no vertex common be special, all its edges in their respective* $(p-1)$-, $(q-1)$-, $(r-1)$- & $(n-p-q-r)$*-spaces x, y, z, t are special and it becomes a 'DoS* $S_{p.q.r.(n-p-q-r+1)}$*' if, and only if, two of the 6 joins yz, zx, xy, xt, yt, zt are perpendicular to the respective complementary opposite spaces of S(A) ; consequently the remaining pair of the 6 joins are also rectangular, and its p-, q-, r-,* $(n-p-q-r+1)$-, $(q+r)$-, $(r+p)$- & $(p+q)$*-altitudes relative respectively to x, y, z, t, yz, zx, xy CONCUR at its 'diorthocentre' D as the 4 altitudes and 3 bialtitudes of the orthogonal tetrahedron, formed by the 4 feet of its p-, q-, r-*

& $(n-p-q-r+1)$-altitudes in x, y, z, t, respectively, perspective with that formed of its p-, q-, r- & $(n-p-q-r+1)$-orthocentres from D as their centre of perspectivity (cf. Art. 10.1).

16.2. Similarly we can define a *DoS* $S_{p.q.r....(n-p-q-r-...+1)}$ and observe that its n' special p-, q-, r-, ... - & $(n-p-q-r-...+1)$-altitudes *concur* at its *di*-orthocentre D as the altitudes of the *associated* orthogonal $(n'-1)$-simplex, formed of their n' feet in its respectively relative n' *special* $(p-1)$-, $(q-1)$-, $(r-1)$-, ... & $(n-p-q-r-...)$-*spaces, perspective with that formed of its* n' p-, q-, r-, ... & $(n-p-q-r-...+1)$-orthocentres from D as their centre of perspectivity (cf. Art. 10.2).

16.3. *The S-point of a DoS* $S_{p.q.r....(n-p-q-r+1)}$ *lies at the centroid of its di-orthocentre and* n'-p-, q-, r-, ..., - & $(n-p-q-r-...+1)$-*orthocentres for multiples* $n'-2$, $p-1$, $q-1$, $r-1$, ..., $(n-p-q-r-...)$ *respectively.* It follows from the relative proposition (Art. 10.3) in regard to a *uni*-orthocentric simplex by the method of induction in the manner we have seen for a *DoS* $S_{q.r.(n-q-r+1)}$, (Art. 12). Thus the S-point of a *DoS* $S_{p.q.r.(n-p-q-r+1)}$ lies at the centroid of its *di*-, p-, q-, r- & $(n-p-q-r+1)$ -orthocentres for multiples 2, $p-1$, $q-1$, $r-1$, $n-p-q-r$.

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Indian Institute of Technology
Kharagpur